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FOUNDATIONS

McKinsey, J. C. C. The decision problem for some classes of sentences without quantifiers. J. Symbolic Logic 8, 61–76 (1943).

The author first proves some general results concerning the decision problem for the open sentences of an extensive class of mathematical systems called algebras. He actually proves the existence of a decision method for the open sentences of some important special types of algebra, including the class of all Abelian groups, of all lattices and of all distributive lattices. His methods are similar to those he previously employed in solving the decision problem for two of C. I. Lewis' systems of strict implication [J. Symbolic Logic 6, 117–134 (1941); these Rev. 3, 290]. An algebra consists of a class of elements together with a finite number of operations on the elements. By a sentence is meant a meaningful expression built up by means of the usual logical connectives from equations involving the operations of the algebra. An open sentence is one without quantifiers. It has the same interpretation as its closure, formed by binding all the variables by universal quantifiers.

The author's most important general result reduces the decision problem for the open sentences of certain classes of algebra to the decision problem for the conditional equations of those classes. It is also shown that a decision method exists for a sentence a of an axiomatizable class A of algebras, provided that a, if false for some algebra of A, is false for a finite algebra of A. Applying this principle to the special class of lattices, it is shown that an open sentence a which is not true for all lattices is false for some particular finite lattice with at most 2^{k+1} elements, where k is the number of distinct terms occurring in a. Likewise, if a is not true for all distributive lattices, then it is false for some particular finite distributive lattice with at most $2^{(2^n)}$ elements, where n is the number of different variables in the sentence a. A decision method is also exhibited for the class of all sentences of a distributive lattice of the form "a implies b," where a and b are closures of open sentences.

O. Frink (State College, Pa.).

Goudge, Thomas A. Science and symbolic logic. Scripta Math. 9, 69–80 (1943). [MF 9016]

This paper consists partly of a popular account of the field of symbolic logic (together with some historical remarks) and partly of argument calculated to show that natural scientists might profitably familiarize themselves with this discipline. The author feels that the introduction of some of the apparatus of symbolic logic into natural science would conduce to clarification and simplification of hypotheses and would permit the use of a rigorous deductive method in fields where ordinary mathematical techniques are employed only with difficulty. *J. C. C. McKinsey.*

Wang, Shianghaw. A system of completely independent axioms for the sequence of natural numbers. J. Symbolic Logic 8, 41–44 (1943).

This paper presents a set of axioms designed to provide a basis for the postulates of Peano for finite cardinal arithmetic. The particular axioms chosen involve no use of either individual constants (that is, constants denoting particular finite cardinals) or functional constants (that is, constants denoting particular one-many relations). Thus "1" and "+" do not figure as primitive symbols. Proofs of the independence of the axioms are given, and their adequacy for Peano's postulates demonstrated. *R. M. Martin.*

Hempel, Carl G. A purely syntactical definition of confirmation. J. Symbolic Logic 8, 122–143 (1943).

This paper presents a definition of the relation of confirmation in syntactical terms (i.e., without referring to states of affairs, or to the behavior of human agents, but solely to the symbols of the object language). The author points out that the paper is only a first step in establishing such a definition, and that further reflection regarding the matter might be desirable in order to get rid of the following two restrictions on his present definition: (1) the definition applies only to sciences which can be formalized in the lower predicate calculus (without identity); and (2) he has defined what it means for a sentence M to confirm a sentence S only in the case where M contains no free or bound variables.

If C is a class of individual constants, and S is a sentence, then the C-development of S is a certain sentence of the object language which means that S is true for the domain of individuals C. (This notion is defined in a precise way, but I lack space here to reproduce it.) If C is the class $\{a, b\}$, for example, and S_1 and S_2 are, respectively, the sentences $(x)P(x)$ and $\sim[(Ex)P(x)]\vee[(x)Q(x)]$, then the C-developments of S_1 and S_2 are, respectively, the sentences $P(a)\cdot P(b)$ and $\sim[P(a)\vee P(b)]\vee[Q(a)\cdot Q(b)]$. In terms of this notion, the relation of direct confirmation is defined in the following way. If M is a sentence without variables, if C is the class of individual constants in M, and if S is any sentence, then M is said to confirm S directly if and only if: (a) the C-development of S is a consequence of M; and (b) if the C-development of S is analytic, then S is itself analytic.

The relation of confirmation is now defined in terms of the relation of direct confirmation as follows. If M is a sentence without variables, and if S is any sentence, then M is said to confirm S if and only if there exists a class K such that: (a) S is a consequence of K; and (b) every member of K is either a consequence of M or is directly confirmed by M. This definition is arrived at, after a preliminary examination of some less satisfactory definitions. The whole discussion is careful and exact; several counter-examples are presented to show the awkwardness of certain possible forms of definition which at first blush might seem intuitively satisfactory. At the end of the paper, arguments are presented in favor of the definition finally arrived at; it is shown, for example, that analytic sentences confirm only analytic sentences. *J. C. C. McKinsey.*

Vaidyanathaswamy, R. On disjunction in intuitionist logic. Proc. Indian Acad. Sci., Sect. A. **17**, 41–45 (1943). [MF 8518]

By the "weak" disjunction $A+B$ of two propositions A and B the author means the absurdity of the simultaneous absurdity of A and B. The "weak" existential quantifier in contexts $(Ex)A(x)$ is to be interpreted as meaning the absurdity of the absurdity of $A(x)$ for all x. By the rules of the formalized intuitionistic logic, it then follows that [Th. 1] weak disjunction is the double negate of the "strong" intuitionistic disjunction, and that the weakly quantified proposition $(Ex)A(x)$ is the double negate of the corresponding "strongly" quantified proposition. The author also notes [Ths. 2 and 3] that conjunction distributes weak disjunction, and that conjunction with a proposition C is distributed by weak disjunction if C is negative. Weak disjunction and weak existential quantification are intended to give essentially the corresponding classical concepts. [For an exact statement of the interrelationship of the classical and intuitionistic concepts, see, however, Glivenko, Acad. Roy. Belgique. Bull. Sci. (5) **15**, 183–188 (1929) and Gödel, Ergebnisse Math. Kolloquium **4**, 34–38 (1933).]

R. M. Martin (Princeton, N. J.).

Goodman, Nelson. On the simplicity of ideas. J. Symbolic Logic **8**, 107–121 (1943).

The author gives a method of assigning to ideas (e.g., of classes and relations of various sorts) numbers or polynomials intended to measure their complexity. The basis is a logic which makes use of the theory of types. A brief word is added as to how this procedure could be extended to systems not making use of the theory of types, but this remark is not very clear to the reviewer. If A is a class of classes, then by a cardinal subclass of A the author means the intersection of A with some cardinal number (a cardinal number being regarded as a class of equivalent classes.) The complexity of an idea of a class of individuals is defined to be 1. Let A be the idea of a class B of classes. If the number of non-empty cardinal subclasses of B is not required to be finite by the conditions imposed on A, then the symbol "∞" is assigned as the index of complexity of A. (The author is in general not concerned with such ideas, however, confining his attention to cases where the complexity is finite.) If the number of non-empty cardinal subclasses of B is limited by the conditions imposed on A merely to be finite (without one's being able to conclude that there are at most some given finite number of non-empty cardinal subclasses) then the symbol "u" is assigned as the index of complexity of A. If, finally, from the conditions imposed on A, we can conclude, for some given finite integer n, that B contains less than $n+1$ non-empty cardinal subclasses, and if we cannot conclude that B contains less than n non-empty cardinal subclasses, then the index of complexity of A is defined to be n. A procedure is sketched for assigning indices of complexity also to entities of higher order (such as classes of classes of classes, etc.).

The index of complexity of a basis (i.e., of a collection of ideas) is defined to be the sum of the indices of the ideas constituting the basis. (Presumably, indices are to be added like ordinary polynomials, though of course the letter "u" as used here is not actually a numerical variable.) Thus the index of complexity of a basis is always either ∞ or a polynomial in u. If $P_1(u)$ is the index of complexity of A_1, and if $P_2(u)$ is the index of complexity of A_2, then A_1 is said to be more complex than A_2 if $P_1(u)$ has the higher coefficient for the highest power of u for which the two polynomials have unequal coefficients. Some space is devoted to a consideration of the relative complexity of sequences as defined by various methods. No attempt is made, however, to apply this notion of complexity to the bases of some of our usual mathematical systems, such as groups, Boolean algebras, and the like; it would seem to the reviewer that most of these systems, at least as usually formulated, would have complexity ∞ according to the author's definition, and hence would not be comparable. *J. C. C. McKinsey.*

Scholz, Heinrich. Leibniz und die mathematische Grundlagenforschung. Jber. Deutsch. Math. Verein. **52**, 217–244 (1942). [MF 9055]

Fraenkel, Abraham Adolf. Problems and methods in modern mathematics. Scripta Math. **9**, 81–84 (1943). [MF 9017]

Miller, G. A. Implications involved in mathematical advances. Science (N.S.) **98**, 38–39 (1943). [MF 8677]

Birkhoff, George D. The mathematical nature of physical theories. American Scientist **31**, 281–310 (1943). [MF 9772]

Whittaker, E. T. The new algebras and their significance for physics and philosophy. Year Book Roy. Soc. Edinburgh **1944**, 10 pp. (1944). [MF 9513]

Milne, E. A. The fundamental concepts of natural philosophy. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 10–24 (1943). [MF 9669]

Greenwood, Thomas. La géométrie de l'univers. Rev. Trimest. Canad. **29**, 295–316 (1943). [MF 8963]

Fantappiè, Luigi. Unified theory of causality and finality in physical and biological phenomena, based on the relativistic wave mechanics. Revista Mat. Hisp.-Amer. (4) **3**, 82–99 (1943). (Spanish) [MF 8949]

Richardson, R. G. D. Applied mathematics and the present crisis. Amer. Math. Monthly **50**, 415–423 (1943). [MF 9090]

ALGEBRA

Kaplansky, Irving. Solution of the "problème des ménages." Bull. Amer. Math. Soc. **49**, 784–785 (1943). [MF 9323]

This famous problem is to find the number of ways n married couples may be seated at a round table, men alternating with women, so that no wife sits next to her own husband. This number is equal to $2n!\lambda(n)$, where $\lambda(n)$ is the number of permutations of $1, 2, \cdots, n$ discordant with both $(1, 2, \cdots, n)$ and $(2, 3, \cdots, n, 1)$. Several solutions of this problem have been given by expressing $\lambda(n)$ in various ways. Recent and more explicit formulas for $\lambda(n)$ have been given by Touchard [C. R. Acad. Sci. Paris **198**, 631–633 (1934)] and Schöbe [Math. Z. **48**, 781–784 (1943); these Rev. **5**, 29] using definite integrals and the "subfactorial"

function, respectively. The present paper gives a formula equivalent to

$$\lambda(n)-(-1)^n 2=(-)^{n-1}n\sum_{r=0}^{n-1}(-1)^r\binom{n+r}{2r+1}r!,$$

which is explicit enough to be called "the solution" of this old problem. The proof of this formula is made very simply by an interesting application of the principle of "cross-classification" of Cayley and Sylvester. *D. H. Lehmer.*

Rado, R. Note on combinatorial analysis. Proc. London Math. Soc. (2) **48**, 122–160 (1943). [MF 9115]

Consider a system of equations

$$(1)\qquad a_{\mu,1}x_1+a_{\mu,2}x_2+\cdots+a_{\mu,n}x_n=b_\mu,\quad 1\leqq\mu\leqq m,$$

where the $a_{\mu,\nu}$, b_μ are complex numbers. Let A be any set of numbers. The system (1) is called regular if for any splitting of A into a finite number of subsets $A_1, A_2, \cdots, A_k$ at least one of these subsets contains a solution of (1). In a previous paper [Math. Z. **36**, 424–480 (1933)] the author determined all systems (1) regular in the set of positive integers.

In the present paper the author gives necessary and sufficient conditions for regularity in the following cases: (i) the $a_{\mu,\nu}$ are arbitrary, $b_\mu=0$, A the set of all numbers different from 0 in a given ring of complex numbers; (ii) the $a_{\mu,\nu}$ and b_μ are arbitrary numbers, A the field of all algebraic numbers; (iii) the $a_{\mu,\nu}$ are algebraic numbers, the b_μ arbitrary, A the field of all complex numbers. In cases (ii) and (iii) the condition for regularity is that for some ξ of A

$$\sum_{\nu=1}^{n}a_{\mu,\nu}\xi=b_\mu,\qquad 1\leqq\mu\leqq m.$$

In case (i) the conditions are slightly more complicated. In proving the results concerning (i) the following result of T. Grünwald is used: given any configuration S consisting of a finite number of lattice points of a Euclidean space, and given a distribution of all lattice points of this space into a finite number of classes, there is at least one class which contains a configuration S' of lattice points which is similar and parallel to S. *P. Erdös* (Lafayette, Ind.).

Dyson, F. J. Three identities in combinatory analysis. J. London Math. Soc. **18**, 35–39 (1943). [MF 9209]

The three identities proved in this paper are due to Rodgers and are similar to the two celebrated identities of Rodgers-Ramanujan except that the "five" has been replaced by "seven." More precisely, the third identity, for instance [which is misprinted], can be written

$$\prod_{n=0}^{\infty}(x_{7n+1}x_{7n+6}x_{7n+7})=(\prod_{n=1}^{\infty}x_{2n})\sum_{m=0}^{\infty}\frac{x_1x_3\cdots x_{2m+1}}{x_2x_4\cdots x_{4m+2}}x^{2m^2+2m},$$

where $x_k=1-x^k$. The present proof is modeled after Hardy's presentation [Ramanujan, Cambridge University Press, Cambridge, England, 1940, pp. 95–99; these Rev. **3**, 71] of one of Rodgers' proofs of the Rodgers-Ramanujan identity. *D. H. Lehmer* (Berkeley, Calif.).

Krishnaswamy Ayyangar, A. A. A general theory of tactical configurations. J. Mysore Univ. Sect. A. **1**, 103–113 (1943). [MF 9354]

A tactical configuration of rank μ is an arrangement of v symbols into b sets of k each such that every r-plet ($r\leqq\mu$) of symbols occurs f_r times. The author shows that $(v-r)/(k-r)=f_r/f_{r+1}$ ($r<\mu$). Generalizing a result of R. A. Fisher [Ann. Eugenics **10**, 52–75 (1940); these Rev. **1**, 348], the author shows that $f_{r+1}\geqq k-r$, $f_r\geqq v-r$ for $0<r<\mu-1$ and derives also other necessary conditions for the existence of a tactical configuration with given characteristics v, k, b, $f_1, \cdots, f_\mu$. Derived residual, complementary and dual configurations are discussed and illustrated by examples. *H. Mann* (Barrytown, N. Y.).

Bose, Raj Chandra. A note on two combinatorial problems having applications in the theory of design of experiments. Science and Culture **8**, 192–193 (1942). [MF 9340]

In a previous paper [Ann. Eugenics **9**, 353–399 (1939); these Rev. **1**, 199] the author gave a method for constructing two-fold triple systems. This method depends on the solution of two combinatorial problems. These problems were first solved by K. N. Bhattacharya [Sankhyā **6**, 313–314 (1943); these Rev. **5**, 29]. The author gives a simpler solution of these two problems. *H. B. Mann.*

Sispánov, Sergio. Algebraic calculation of the regular 11-gon. Revista Union Mat. Argentina **9**, 77–88 (1943). (Spanish) [MF 9439]

Solution of the cyclotomic equation $(x^{11}-1)/(x-1)=0$. *P. Scherk* (Saskatoon, Sask.).

Bellman, Richard. A note on determinants and Hadamard's inequality. Amer. Math. Monthly **50**, 550–551 (1943). [MF 9532]

Laguardia, Rafael. On certain systems of linear equations and their determinants. Math. Notae **3**, 129–141 (1943). (Spanish) [MF 9561]

Niven, Ivan. Two observations concerning algebraic equations for matrices. Amer. J. Math. **65**, 660–662 (1943). [MF 9392]

Let A be a matrix with elements in an algebraically closed field k and with minimum function $\psi(z)=\prod_{i=1}^{s}(z-\alpha_i)^{m_i}$. If $P(w,z)$ is a polynomial of two variables w,z in k which actually contains w, the equation $P(w,z)=0$ defines an algebraic function $w(z)$ and $\varphi(z)=\beta+\beta_1(z-\alpha)+\cdots+\beta_{k-1}(z-\alpha)^{k-1}$ is a branch at $z=\alpha$ which is regular up to the kth order if $P(\varphi(z),z)\equiv 0 \pmod{(z-\alpha)^k}$. If, for each $i=1, \cdots, r$, there is a branch $\varphi_i(z)$ at $z=\alpha_i$ which is regular up to the order m_i, then the simultaneous congruences $f(z)\equiv\varphi_i(z) \pmod{(z-\alpha_i)^{m_i}}$ ($i=1, \cdots, r$) uniquely determine a solution $W=f(A)$ of the equation $P(W,A)=0$. Every solution W which is expressible as a polynomial in A may be obtained in this manner. *N. H. McCoy.*

Rutherford, D. E. On the matrix representation of complex symbols. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 25–27 (1943). [MF 9667]

It was shown by Temple that the Clebsch-Aronhold symbol $(a_1x_1+\cdots+a_nx_n)^m$ for a homogeneous form of degree m in the variables can be represented as a product of matrices involving these variables. Here the author extends the method to Weitzenböck's complex symbols in terms of which the generalised Plücker coordinates of an $m-1$ dimensional space lying in an $n-1$ dimensional one can be expressed. *G. de B. Robinson* (Ottawa, Ont.).

Abstract Algebra

***van der Waerden, B. L. Moderne Algebra. Parts I and II.** G. E. Stechert and Co., New York, 1943. 272+224 pp. $6.00.

Reprint of the second revised German edition of 1937–1940.

Ore, Oystein. Chains in partially ordered sets. Bull. Amer. Math. Soc. **49**, 558–566 (1943). [MF 8856]

The following generalization of the Jordan-Hölder theorem to arbitrary partially ordered sets is obtained. Let P be a partially ordered set in which a chain between two elements is always finite. All maximal chains between fixed endpoints have the same length if and only if this is true for all pairs of chains constituting "simple cycles" [cf. the following review]. A sufficient condition is the "quadrilateral condition": when a and b are distinct elements prime over d, then there shall exist at least one element m prime over a and b. *G. Birkhoff* (Cambridge, Mass.).

MacLane, Saunders. A conjecture of Ore on chains in partially ordered sets. Bull. Amer. Math. Soc. **49**, 567–568 (1943). [MF 8857]

A "simple deformation" of a maximal chain C: $a_0<a_1<\cdots<a_n$ of a partially ordered set consists in replacing some interval C'': $a_i<a_{i+1}<\cdots<a_{i+j}$ of C by C': $a_i<a'_{i+1}<\cdots<a'_{i+k}=a_{i+j}$, where $a'_{i+k-1}\cap a_{i+j-1}=a_i$ and $a_{i+1}\cup a'_{i+1}=a_{i+j}$. Here C'' and C' are said to form a "simple cycle." MacLane proves that, if P is a partially ordered set in which every chain joining two elements is finite, then any maximal chain between two elements $b<a$ can be deformed into any other by a succession of simple deformations. *G. Birkhoff* (Cambridge, Mass.).

Baer, Reinhold. Radical ideals.. Amer. J. Math. **65**, 537–568 (1943). [MF 9385]

The author defines a radical ideal I in the ring R to be a two-sided nilideal such that R/I contains no nilpotent ideals. In general there are many radical ideals, but their union U and intersection L are again radical ideals called the upper and lower radical, respectively. A surprising example shows that an ideal between L and U is not necessarily a radical ideal.[In this connection it might be noted that Levitzki's version of the radical [Bull. Amer. Math. Soc. **49**, 462–466 (1943); these Rev. **4**, 238] is a radical ideal in general distinct from L or U.] Many of the author's results are proved using the following weakened form of the descending chain condition: every right ideal contains a nonzero minimal right ideal. Thus if R/L satisfies this condition, $L=U=$ the radical K; and if every quotient ring of R does, some (possibly transfinite) power of K is 0. Again, if R/U satisfies the weakened minimal condition and if R has a left-identity, then U is the intersection of the maximal right ideals in R. This suggests the dual concept of the union of the minimal right ideals (if any), which the author calls the anti-radical M [perhaps better the right anti-radical?]. Under suitable conditions M consists precisely of all left annihilators of U. Defining M_2 so that M_2/M is the anti-radical of R/M, etc., one obtains a transfinite chain $\{R_m\}$, and $R_m=R$ for some ordinal m if and only if every quotient ring of R satisfies the weakened minimal condition. The preceding theory is applied to obtain various connections between the maximal and minimal conditions and the existence of an identity; these provide generalizations of results of Hall [Ann. of Math. (2) **40**, 360–369 (1939)] and Hopkins [Ann. of Math. (2) **40**, 712–730 (1939); these Rev. **1**, 2]. Two other recent contributions to the theory of algebras are generalized to rings, again with various suitable minimal conditions: Hall's theorem on algebras bound to their radical [Trans. Amer. Math. Soc. **48**, 391–404 (1940); these Rev. **2**, 122] and Perlis' characterization of the radical [Bull. Amer. Math. Soc. **48**, 128–132 (1942); these Rev. **3**, 264]. Some readers may join the reviewer in wondering whether all the various forms of the minimal condition used by the author are distinct. *I. Kaplansky.*

Baer, Reinhold. Rings with duals. Amer. J. Math. **65**, 569–584 (1943). [MF 9386]

Two rings R, R' are right-duals if there exists a duality (that is, a 1-1 correspondence which inverts order) of the partially ordered sets of their right ideals. If R' has a left-identity, then any right ideal of R' is contained in a maximal right ideal and the dual property will hold in R. If it is further assumed that every quotient ring of R has a right-dual with a left-identity and that R has an identity, the results obtained by the author in the preceding paper enable him to prove that R has the minimal condition on right ideals and that R is uni-serial in the following sense: if J is a right ideal not in the radical, while all right ideals contained in J are in the radical, then the right ideals contained in J form a linearly ordered set. The remainder of the paper is devoted to the study of the possible dualities of uni-serial primary rings, a primary ring being the set of matrices E_n over a ring E, where E modulo its radical P is a division ring. The following results are obtained: (1) E and E_n are uni-serial if and only if every right ideal in E is a power of P; (2) $E_n(n>1)$ possesses a right-dual if and only if every left ideal in E is likewise a power of P; (3) primarity and uni-seriality are preserved under duality; (4) for $n>2$ and right-dual of E_n is anti-isomorphic to E_n. *I. Kaplansky* (Cambridge, Mass.).

Birkhoff, Garrett. The radical of a group with operators. Bull. Amer. Math. Soc. **49**, 751–753 (1943). [MF 9317]

Let G be a group, Ω a set of endomorphisms that contains the inner automorphisms and $\mathfrak{s}$ a class of Ω-groups that is invariant under isomorphism. The term simple group is used by the author for a group G which is not in $\mathfrak{s}$ and which contains no proper subgroup invariant under the elements of Ω. A semi-simple group is a group that is a direct sum of simple groups. The radical R of G is defined to be the intersection of all Ω-subgroups S such that G/S is simple. It is proved that G/S is semi-simple if and only if S contains R. This is an extension to groups of a result of Albert on the radical of a nonassociative algebra [Bull. Amer. Math. Soc. **48**, 891–897 (1942); these Rev. **4**, 130]. *N. Jacobson.*

Schilling, O. F. G. Normal extensions of relatively complete fields. Amer. J. Math. **65**, 309–334 (1943). [MF 8206]

A field F with a valuation is called relatively complete if Hensel's lemma is valid. In this paper the author develops a Hilbert theory for normal extension L of F under the assumption that the residue field $\mathfrak{F}$ of F is perfect. He calls a separable extension L of F unramified if $(K:F)=(\mathfrak{K}:\mathfrak{F})$ for any finite subfield K of L and the corresponding residue field $\mathfrak{K}$. It is shown that there exists a maximal unramified subfield, the inertial field L_I of L, whose residue field coincides with that of L. A ramification field L_R containing L_I

is defined; L_R is an Abelian extension of L_I and is determined by the structure of the value group Γ of L. The author investigates the possibility of constructing normal extensions L of F with pre-assigned residue field $\mathfrak{L}$ and value group Δ. The results are complete for the case where the residue field $\mathfrak{F}$ is finite. A survey of all possible Galois groups G of the normal extensions L of F is given by determining an abstract discrete group that acts as a universal covering group for all the G. The universal group is completely determined by F and the value group Δ of F.

N. Jacobson (Baltimore, Md.).

Wade, L. I. Certain quantities transcendental over $GF(p^n, x)$. II. Duke Math. J. **10**, 587–594 (1943). [MF 8983]

Let $GF(q)$ denote a fixed finite field of order $q=p^n$, $GF[q, x]$ the ring of polynomials in the indeterminate x over $GF(q)$ and $GF(q, x)$ the quotient field of $GF[q, x]$. For a positive integer k and indeterminate t, $[k]=x^{q^k}-x$, $F_k=[k][k-1]^q\cdots[1]^{q^{k-1}}$, $L_k=[k]\cdots[1]$, $L_0=F_0=1$;

$$\Psi(t)=\sum_{j=0}^{\infty}(-1)^j(t^{q^j}/F_j).$$

It is shown that $\sum_{j=0}^{\infty}(1/L_j^{\gamma})$, where γ is any positive rational integer, is transcendental over $GF(q, x)$. This yields a new proof of the transcendence of ξ, where $\xi\neq 0$ belongs to a super-field over $GF(q, x)$ and has the property that $\Psi(E\xi)=0$ for every element E of $GF[q, x]$. The author had used other methods to prove the transcendence of ξ and other quantities in an earlier paper of like title [Duke Math. J. **8**, 701–720 (1941); these Rev. **3**, 263].

J. L. Dorroh (Baton Rouge, La.).

Nesbitt, C. and Scott, W. M. Some remarks on algebras over an algebraically closed field. Ann. of Math. (2) **44**, 534–553 (1943). [MF 8878]

A semi-primitive ring S is a ring with radical N for which the residue class ring of R modulo N is a direct sum of division rings $D_1, D_2, \cdots, D_r$. Every ring R which satisfies the minimum condition for left-ideals can be completely described by means of a certain semi-primitive subring S and a number of positive integers $f_1, f_2, \cdots, f_r$. We may call S the basic ring of R. The authors consider the case that R is an algebra with unit element over an algebraically closed field K. They give a construction of the semi-primitive algebra S and study the connection with the Cartan basic system [cf. C. Nesbitt, Ann. of Math. (2) **39**, 634–658 (1938)]. It is shown that there exists a (1-1) correspondence between the representations of R and those of its basic algebra S; corresponding representations have the same structure. The basic algebra of a direct product $R_1\times R_2$ is the direct product $S_1\times S_2$ of the basic algebras S_i of R_i. The commutator algebra of a representation of R and its basic algebra are studied. The second part of the paper deals with symmetric functions of R, that is, linear homogeneous functions $\varphi(\alpha)$, defined for α in R with values in K, for which $\varphi(\alpha\beta)=\varphi(\beta\alpha)$ for all α, β in R. For instance, the characters of the irreducible representations form such symmetric functions. A (1-1)-correspondence between the symmetric functions of R and those of its basic algebra S is established. The number q of linearly independent symmetric functions is at most equal to the sum $s=\sum c_{ii}$ of the Cartan invariants of unmixed type; a number of questions related to this result are investigated. An algebra R is symmetric [cf. R. Brauer and C. Nesbitt, Proc. Nat. Acad. Sci. U.S.A. **23**, 430–434 (1937)] if and only if its basic algebra is symmetric. In the last part of the paper the regular representations are written in terms of the elementary modules [cf. W. M. Scott, Ann. of Math. (2) **43**, 147–160 (1942); these Rev. **3**, 263].

R. Brauer.

Hillman, A. P. A note on differential polynomials. Bull. Amer. Math. Soc. **49**, 711–712 (1943). [MF 9308]

The identity $C_0F+C_1F_1+\cdots+C_sF_s=0$, where F is any nonzero differential polynomial, F_i is its ith derivative and the C_i are differential polynomials, is shown to imply that the coefficients C_i are in the perfect differential ideal of F. The same result is stated for the coefficients of identically vanishing homogeneous polynomials in the F_i, of any degree.

H. Levi (New York, N. Y.).

NUMBER THEORY

Sispánov, Sergio. A problem in Diophantine analysis. Revista Union Mat. Argentina **9**, 41–48 (1943). (Spanish) [MF 9262]

The author solves the Diophantine equation

$$x^3+y^3+z^3-3xyz=kt^3 \tag{1}$$

in the following way. Let $(x-z, y-z, t)=r$. Hence

$$x=z-ar,\quad y=z-br,\quad t=cr, \tag{2}$$

where $(a, b, c)=1$, Put

$$\frac{r}{3(a^2-ab+b^2)}=\frac{h}{d},\qquad (h, d)=1. \tag{3}$$

Then

$$z=\frac{r(kc^3+a^3+b^3)}{3(a^2-ab+b^2)}=\frac{h}{d}(kc^3+a^3+b^3). \tag{4}$$

Similar formulas follow from (2), (3) and (4) for x, y, t. We have from (3) and (4)

$$d\mid d'=(kc^3+a^3+b^3,\ 3(a^2-ab+b^2)).$$

If we drop the assumption $(h, d)=1$, we may put $d=d'$. Then we obtain the general solution of (1) depending upon the four parameters a, b, c, h.

P. Scherk.

Dyer, P. S. A solution of $A^4+B^4=C^4+D^4$. J. London Math. Soc. **18**, 2–4 (1943). [MF 9203]

Euler gave solutions of the Diophantine equation $A^4+B^4=C^4+D^4$. A simpler solution was obtained by Gérardin [Intermédiaire des Math. **24**, 51 (1917)]. In both solutions A, B, C, D are homogeneous polynomials in two variables of degree seven with integral coefficients. The author gives a very simple proof for Gérardin's result.

A. Brauer.

Rosenthall, E. Diophantine equations reducible in bi-quadratic fields. Duke Math. J. **10**, 463–470 (1943). [MF 8967]

This paper gives the "complete solution in rational integers" of the Diophantine equations $x^3+y^3=u^2+v^2$ and $x^4+y^4=u^2+v^2$. The method, which is applicable to equations completely reducible in the special bi-quadratic field produced by the adjunction of $a^{\frac{1}{2}}$ and $b^{\frac{1}{2}}$ to the rationals, consists in the application of the fundamental lemma to the

effect that all integer solutions of $XY=ZW$ are given by $X=AB$, $Y=CD$, $Z=AD$, $W=CB$, where A, B, C, D are arbitrary integers with B, D coprime. The solutions of the two equations given above are far from explicit since they are expressed in terms of parameters which are not rational integers and in terms of solutions of linear equations in bi-quadratic fields. The author shows how these linear equations can be reduced to the solution of linear equations in rational integers. The complete solutions of the equations originally proposed, in terms of rational integer parameters, could be obtained from results of this paper.

D. H. Lehmer (Berkeley, Calif.).

Rosenthall, E. On some cubic Diophantine equations. Amer. J. Math. **65**, 663–672 (1943). [MF 9393]

The cubic equations considered in this paper are $x^3+y^3=z^3+w^3$, $x^3+y^3=z^2$, $x^3+y^3+z^3-3xyz=u^3+v^3+w^3-3uvw$. These equations are written in multiplicative form by the use of integers of the form $a+b\rho$, where ρ is an imaginary cube root of unity, and are then completely solved by applying the lemma to the effect that all solutions, in integers of the above type, of the equation $XY=ZW$ are given by $X=US$, $Y=VT$, $Z=UT$, $W=VS$. Unfortunately the solutions are far from explicit since they are given in terms of parameters which are not rational integers but integers of the above-mentioned quadratic field. Rules have to be given for selecting these integers so that the resulting values of $x, y, z, \cdots$ are rational integers [see also the paper reviewed above].

D. H. Lehmer.

Brauer, Alfred. On the non-existence of odd perfect numbers of form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$. Bull. Amer. Math. Soc. **49**, 712–718 (1943). [MF 9309]

It was shown by Euler that an odd perfect number, if there is one, must have the form $n=p^\alpha q_1^{2\beta_1}\cdots q_t^{2\beta_t}$, where p, q_i are primes and $p\equiv\alpha\equiv 1 \pmod 4$. Recently it was shown by Steuerwald that $\beta_1=\beta_2=\cdots=\beta_t=1$ is impossible [S.-B. Bayer. Akad. Wiss. **1937**, 68–72] and by Kanold that the same is true for $\beta_1=\beta_2=\cdots=\beta_{t-1}=\beta_t=2$ [J. Reine Angew. Math. **183**, 98–109 (1941); these Rev. **3**, 268]. The author proves that $\beta_1=\beta_2=\cdots=\beta_{t-1}=1$, $\beta_t=2$ is also impossible. The proof depends on two lemmas of which the following is one. If q is a positive prime then q^2+q+1 cannot be the mth power of an integer ($m>1$). The proof of this lemma uses a theorem of Nagell to the effect that the Diophantine equation $x^2+x+1=y^m$ has no solution with $y\neq\pm 1$ if m is not a power of 3 and $m>1$. The second lemma is similar and uses another similar result of Nagell.

H. W. Brinkmann (Swarthmore, Pa.).

Brauer, Alfred. Note on the non-existence of odd perfect numbers of form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$. Bull. Amer. Math. Soc. **49**, 937 (1943). [MF 9694]

Statements concerning the similarity and the mutual independence of the proofs given by Kanold [J. Reine Angew. Math. **184**, 116–123 (1942); these Rev. **5**, 33] and the author [cf. the preceding review].

Oblath, Richard. On products of consecutive integers. Revista Mat. Hisp.-Amer. (4) **2**, 190–210, 253–270 (1942). (Spanish) [MF 8350]

This paper contains various contributions to the conjecture that the product

$$P_k(x)=x(x+1)\cdots(x+k-1)$$

of k consecutive integers ($k>1$) cannot be a perfect nth power ($n>1$). For example, the author proves that P_3 cannot be an nth power if n is a prime for which $2^{n-1}\neq 1 \pmod{n^2}$; this proves the conjecture for P_3 if n(prime) <16000 with the possible exceptions of $n=1093$, 3511. Similar results are obtained for P_4, P_5, P_7. Again it is proved that $P_k(x)=y^n$ can have at most one solution (for given n); similar results are given for P_4, P_5, P_7. All of these special results have been superseded by the proof of the conjecture for $k\leqq 9$ which has been given by G. Szekeres. This result has apparently not been published but was communicated to the author.

Some of the lemmas used in the proofs are themselves of interest. For example, it is proved that the equations $x^n\pm 1=2^{n-2}y^n$ have no solutions in positive integers if n is a prime for which $2^{n-1}\neq 1 \pmod{n^2}$. In the theorem concerning the number of solutions a theorem of Siegel is used to establish the uniqueness. A few independent results may be quoted: (1) P_k cannot be a cube or a fifth power if $k\leqq 23$; (2) $P_k(x)$ cannot be an nth power if $k>\frac{1}{2}\sqrt{x}$; (3) $P_k(x)$ can be a perfect cube for at most $3^{2k/(\log k)}$ values of x; (4) $P_k(x)$ can be an nth power ($n\geqq 5$) for at most $2^{(7/2)k+6\sqrt{k}}$ values of x.

H. W. Brinkmann.

Bell, E. T. Algebraic identities in the theory of numbers. Amer. Math. Monthly **50**, 535–541 (1943). [MF 9528]

This paper is concerned with identities which express numbers in an arithmetical progression as sums of like powers of linear functions, as, for example, the identity

$$6m+3=4(m+1)^3-(2m+1)^3+4m^3,$$

from which it follows that every integer is of the form

$$x^3+y^3+4(z^3+w^3),$$

the letters being integers $\geqq 0$. Such identities have an extensive history in connection with the problem of representing numbers as sums of like powers. The author gives some very general identities of this kind. These are derived from the binomial expansions

$$\sum_{r=1}^{n-1}\binom{n}{r}a_i^{n-r}b_i^r x^{n-r}y^r\equiv(a_ix+b_iy)^n-a_i^nx^n-b_i^ny^n,$$

where the a's and b's are integer parameters. Solving this system of $n-1$ equations for $x^{n-1}y=N$, it is clear that, unless its determinant $D=0$, DN can be expressed as a sum of nth powers. Other identities follow by differentiation and other devices. Many special cases are exhibited, including some giving sets of integers having equal sums of sth powers for $s=0, 1, \cdots, n-2$, as, for example, $(x+5)^s+5(x+3)^s+10(x+1)^s=(x-5)^s+5(x-3)^s+10(x-1)^s$, $s=0, 1, 2, 3, 4$. Solutions involving three parameters are given of such Diophantine equations as

$$x_1^3+x_2^3+x_3^{3n}+3x_4^{3n}=y_1^3+y_2^3+y_3^{3n}+3y_4^{3n}$$

for every integer $n>0$. Applications in which the parameters of the identities are algebraic are mentioned.

D. H. Lehmer (Berkeley, Calif.).

Mills, W. H. Iteration of the ϕ function. Amer. Math. Monthly **50**, 547–549 (1943). [MF 9531]

In a recent paper of H. Shapiro [same Monthly **50**, 18–30 (1943); these Rev. **4**, 188] it was conjectured that for every n the least solution x_n of $\phi^n(x)=2$ (where $\phi^n(x)=\phi^{n-1}(\phi(x))$ is the nth iterate of Euler's totient function ϕ) is a prime. The first ten such x's are given in the above-mentioned review and are primes. The present paper points out that $x_{11}=17\cdot 137$. A detailed proof is given of the fact that $x_{12}=(2^{16}+1)^2$.

D. H. Lehmer (Berkeley, Calif.).

Kesava Menon, P. On arithmetic functions. Proc. Indian Acad. Sci., Sect. A. **18**, 88–99 (1943). [MF 9330]

Generalization and extensions of results of R. Vaidyanathaswamy on multiplicative arithmetic functions [Trans. Amer. Math. Soc. **33**, 579–662 (1931)]. In particular, the author generalizes the operations of "composition" and "compounding." As an application he evaluates certain multiple Dirichlet and Lambert series. *L. Carlitz.*

Dörge, Karl. Beweis des Reziprozitätsgesetzes für quadratische Reste. Math. Ann. **118**, 310–311 (1942). [MF 8932]

A proof of the quadratic reciprocity theorem based on Gauss's lemma, which is practically equivalent to that of Frobenius [S.-B. Berlin. Akad. Wiss. **1914**, 335–349]. *H. W. Brinkmann* (Swarthmore, Pa.).

Aigner, Alexander and Reichardt, Hans. Stufenreihen im Potenzrestcharakter. J. Reine Angew. Math. **184**, 158–160 (1942). [MF 9036]

Let p be a prime of the form $4n+1$, so that it can be represented by x^2+4u^2. It is possible to distinguish between the cases $u=t$ (odd), $u=2t$, $u=4t$, $u=8t$, according to the character of -2 modulo a prime ideal factor $\mathfrak{p}$ of p in the field $R(\zeta)$, where ζ is a primitive 8th root of unity. In fact, the cases mentioned correspond, respectively, to -2 being a quadratic nonresidue, quadratic residue, quartic residue, octic residue, modulo $\mathfrak{p}$. In the present paper the authors start with a prime of the form $8n+1$ which can be always represented by x^2+16u^2. Here $u=t$, $2t$, $4t$, $8t$ according as the number $4+3\sqrt{2}$ (which is in $R(\zeta)$) is a quadratic nonresidue, quadratic residue, quartic residue, octic residue, modulo $\mathfrak{p}$. Such a prime can also be represented by x^2+8u^2 and here the number $2+2\sqrt{2}$ serves to distinguish the four cases mentioned in exactly the same way. *H. W. Brinkmann* (Swarthmore, Pa.).

Perron, Oskar. Über die Berechnung der Grundeinheit in reellen quadratischen Körpern und Ringen. J. Reine Angew. Math. **185**, 106–110 (1943). [MF 9592]

This paper is occasioned by the appearance of a table [Wilhelm Patz, Tafel der regelmässigen Kettenbrüche für die Quadratwurzeln aus den natürlichen Zahlen von 1–10000, Leipzig, 1941] unknown to the reviewer, giving only the partial quotients b_ν in the first half of the period of the continued fraction

$$D^{\frac{1}{2}}=b_0+\frac{1}{b_1+}\,\frac{1}{b_2+}\cdots.$$

Some problems, however, demand the knowledge of the denominators Q_ν of the complete quotients $(P_\nu+D^{\frac{1}{2}})/Q_\nu=b_\nu+\cdots$. In particular, problems having to do with the automorphs of indefinite binary quadratic forms or the units of the quadratic field $K(D^{\frac{1}{2}})$ depend on the existence of a $Q_\nu=4$. Hence the author of this paper offers the following theorem: If Q_ν has a value c, then the partial quotient b_ν lies in the interval

$$\frac{2b_0+2}{c}-2\leqq b_\nu\leqq\frac{2b_0}{c}\qquad(b_0=[D^{\frac{1}{2}}])$$

and, if $D>c^4$, the converse is true: if b_ν lies in this interval then $Q_\nu=c$. In the interesting case $c=4$, to look for a $Q=4$ we may look instead for $ab=[b_0/2]$ or $[b_0/2]-1$. *D. H. Lehmer* (Berkeley, Calif.).

Banerjee, D. P. On the solution of the "easier" Waring problem. Bull. Calcutta Math. Soc. **34**, 197–199 (1942). [MF 9002]

In the "easier" Waring problem, $v(k)$ is defined as the least value of s for which every integer n is representable as $\pm x_1^k\pm\cdots\pm x_s^k$ with integral $x_1, \cdots, x_s$ and with signs which may depend on n. Fuchs and Wright [Quart. J. Math., Oxford Ser. **10**, 190–209 (1939); these Rev. **1**, 69] obtained upper and lower bounds for $v(k)$ for $k\leqq 20$, and the present paper extends their results to $21\leqq k\leqq 30$. *H. Davenport* (Bangor, Wales).

Gupta, Hansraj. On the class-numbers of binary quadratic forms. Univ. Nac. Tucumán. Revista A. **3**, 283–299 (1942). [MF 9271]

This paper contains a 14 page table giving for each possible value of $d=4ac-b^2\leqq 12500$ the number $h'(d)$ of properly primitive classes of forms $ax^2+bxy+cy^2$ of determinant $-d$. The table is in two parts devoted, respectively, to the cases $d=4n$, $4n+3$. The total number $h(d)$ of classes can be found by summing the values of $h'(d/k^2)$ over all the square divisors k^2 of d. This is the most extensive table of its kind ever published and should be of considerable future use in investigating the properties of this interesting function. *D. H. Lehmer* (Berkeley, Calif.).

Tietze, Heinrich. Über die Anzahl der Lösungen gewisser Aufgaben über Gitterpunktfiguren. Math. Ann. **118**, 290–298 (1942). [MF 8929]

This paper is a continuation of the author's series of papers on the n-dimensional graphical representation of partitions [Monatsh. Math. Phys. **49**, 1–52 (1940); S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1940**, 23–54 (1940); these Rev. **1**, 323; **3**, 166]. Let $\mathfrak{M}$ be any set of m lattice points in n-dimensions $(x_1, x_2, \cdots, x_n)$ with positive integer coordinates. This set of points gives rise to n conjugate partitions $A^{(\nu)}=A^{(\nu)}(\mathfrak{M})$ $(\nu=1, 2, \cdots, n)$ of m in which the μth part of $A^{(\nu)}$ is the number of lattice points of $\mathfrak{M}$ lying on the $(n-1)$-dimensional hyperplane $x_\nu=\mu$. The converse question considered in this paper is as follows: under what circumstances does a given set $A^{(\nu)}$ of n partitions, all of the same number m, determine a lattice point set $\mathfrak{M}$ belonging to it in the above sense? In treating this question the author uses the notion of a reduced (or compressed) lattice point set, defined as a set $\mathfrak{R}$ of lattice points such that if $(x_1', x_2', \cdots, x_n')$ belongs to $\mathfrak{R}$ so also do those points $(x_1, x_2, \cdots, x_n)$ for which $0<x_\nu\leqq x_\nu'$ $(\nu=1, 2, \cdots, n)$. Two partitions $A=(a_1, a_2, \cdots, a_k)$ and $B=(b_1, b_2, \cdots, b_k)$ of the same number (with the parts nonincreasing) being given, we define $A\geqq B$ to mean that, for each $\mu=1, 2, \cdots, k$, $\sum_{\lambda=1}^{\mu}a_\lambda\geqq\sum_{\lambda=1}^{\mu}b_\lambda$. The two results of the paper are then as follows. The n given partitions $A^{(\nu)}$ are the n conjugate partitions of a point set $\mathfrak{M}$ if and only if there exists a reduced lattice point set $\mathfrak{R}$ such that $A^{(\nu)}(\mathfrak{R})\geqq A^{(\nu)}$ $(\nu=1, 2, \cdots, n)$. If this set $\mathfrak{M}$ exists uniquely it is reduced. *D. H. Lehmer* (Berkeley, Calif.).

Tietze, Heinrich. Komprimierte Gitterpunktmengen und eine additiv-zahlentheoretische Aufgabe. J. Reine Angew. Math. **184**, 49–64 (1942). [MF 9029]

This paper is a continuation of the author's investigations of n-dimensional lattices and associated partitions [see the preceding review]. The main result of the paper is concerned with the question of the number of different reduced lattices of n dimensions that belong, in the sense of the

preceding review, to a given set of n partitions of the same number m. The author shows by a general example that even when $n=3$ there exist, for each integer h, three partitions of the number $m=4h^3+9h^2$ with which there are associated at least 2^h different reduced lattices.

D. H. Lehmer (Berkeley, Calif.).

Bell, E. T. Interpolated denumerants and Lambert series. Amer. J. Math. 65, 382–386 (1943). [MF 8696]

Let $D(n)=D(n|a_1, a_2, \cdots, a_r)$ be the number of sets $(x_1, \cdots, x_r)$ of nonnegative integers x such that $a_1x_1+\cdots+a_rx_r=n$. This numerical function was called by Sylvester the denumerant of n with respect to the positive integers $a_1, \cdots, a_r$. Sylvester and Cayley showed that $D(n)=A(n)+U(n)$, where $A(n)$ is a polynomial in n of degree $r-1$ and $U(n)$ is a periodic function of period a, the least common multiple of $a_1, \cdots, a_r$. This paper gives an elementary proof of the fact that, for any fixed b, $D(am+b)$ is a polynomial in m of degree $r-1$. Hence, if we compute once for all the values of $D(aj+b_j)$, $0\leqq b_j<a$; $j=1, \cdots, r$, and regard these as known constants, an explicit formula for $D(am+b)$ may be found by the straightforward application of Lagrange's interpolation formula. As an application the author considers the problem of expanding in powers of x the function $L(x)$ defined by the r-fold Lambert series

$$L(x)=\sum_{n=1}^{\infty}\frac{f(n)x^{ns}}{(1-x^{a_1n})\cdots(1-x^{a_rn})},$$

where $s=a_1+\cdots+a_r$. *D. H. Lehmer* (Berkeley, Calif.).

Dyson, F. J. On the order of magnitude of the partial quotients of a continued fraction. J. London Math. Soc. 18, 40–43 (1943). [MF 9210]

Let ξ be any irrational in the range $0<\xi<1$; consider the regular continued fraction of ξ: $[1/a_n]_1^\infty=\xi$. Given any nondecreasing positive function $\varphi(n)$, write

$$b_n=\max_{\nu\leq n}\frac{\varphi(n)a_\nu}{\varphi(\nu)}.$$

The following theorem is proved: if $\sum 1/\varphi(n)=\infty$, then $\sum b_n^{-1}=\infty$ for almost all ξ. The author remarks that, in the special case $\varphi(n)=1$, Khintchine obtained a sharper result. *O. Szász* (Cincinnati, Ohio).

Krishnaswami Ayyangar, A. A. Theory of the nearest square continued fraction. J. Mysore Univ. Sect. A. 1, 21–32 (1940). [MF 9353]

Krishnaswami Ayyangar, A. A. Theory of the nearest square continued fraction. J. Mysore Univ. Sect. A. 1, 97–117 (1941). [MF 9355]

These papers are concerned with a peculiar semi-regular continued fraction algorithm somewhat akin to the nearest integer algorithm. The expansion is defined only for the standard quadratic surd $(P+\sqrt{R})/Q$ and is presumably based on Bhaskara's "cyclic method" of solving the Pell equation, although the author does not trace the connection in detail. The discussion of the algorithm is made ab initio, very little being borrowed from the well-established theory of semi-regular continued fractions.

More precisely, the algorithm proceeds as follows. Let the greatest integer in the given surd $(P+\sqrt{R})/Q$ be a and let

$$\xi_0=\frac{P+\sqrt{R}}{Q}=a+\frac{Q'}{P'+\sqrt{R}}=a+1-\frac{Q''}{P''+\sqrt{R}}.$$

Then the first partial quotient b_0 in the expansion of ξ_0 is taken as a if $|Q'|<|Q''|$ or, in case $|Q'|=|Q''|$, if $Q<0$. Otherwise b_0 is taken as $a+1$. This is the same as choosing a if P'^2 is closer to R than P''^2, hence the author's name for the algorithm. The second complete quotient is taken as $\xi_1=(P'+\sqrt{R})/Q'$ or $(P''+\sqrt{R})/Q''$ according as $b_0=a$ or $a+1$, and the process repeated. For example, the expansion of $\xi_0=\sqrt{21}$ by this algorithm is

$$\sqrt{21}=5-\dot{\frac{1}{2}}+\dot{\frac{1}{2}}+\dot{\frac{1}{2}}-\dot{\frac{1}{10}}-\dot{\frac{1}{2}}+\cdots.$$

The nearest integer algorithm gives

$$\sqrt{21}=5-\dot{\frac{1}{2}}+\dot{\frac{1}{3}}-\dot{\frac{1}{2}}+\dot{\frac{1}{9}}-\dot{\frac{1}{2}}+\cdots.$$

The regular algorithm gives

$$\sqrt{21}=4+\dot{\frac{1}{1}}+\dot{\frac{1}{1}}+\dot{\frac{1}{2}}+\dot{\frac{1}{1}}+\dot{\frac{1}{1}}+\dot{\frac{1}{8}}+\dot{\frac{1}{1}}+\cdots.$$

Pages 115–116 contains a table of expansions of $\sqrt{R}$ for all nonsquare integers $R<100$. Out of these 90 cases 75 of the expansions are the same as those of the nearest integer algorithm. The partial quotients of a period in the expansion of $\sqrt{R}$ are not always symmetric as above. For instance, the algorithm gives

$$\sqrt{97}=10-\dot{\frac{1}{7}}-\dot{\frac{1}{3}}-\dot{\frac{1}{2}}+\dot{\frac{1}{2}}-\dot{\frac{1}{7}}-\dot{\frac{1}{20}}-\dot{\frac{1}{7}}-\cdots.$$

The author proves that the period is either symmetric, as in the regular case, or else "almost symmetric," that is, symmetric except for three central asymmetric partial quotients of the form b_n, 2, b_n-1. In 85 of the 90 cases mentioned above, there is pure symmetry, however.

The theory of this continued fraction in most respects closely parallels the classical regular case of Lagrange. The slight blemishes characteristic of the new (or should one say 12th century) algorithm do not seem to be compensated by any useful features other than that possessed by the nearest integer algorithm. Further investigations of more difficult problems are promised. *D. H. Lehmer.*

ANALYSIS

Theory of Functions of Complex Variables

Montel, Paul. Sur les rapports entre les dérivées et les différences divisées. Mathematica, Timişoara 19, 1–11 (1943). [MF 9426]

Let $d_1f(z_0)=\{f(z_1)-f(z_0)\}/\{z_1-z_0\}$ be the divided difference of $f(z)$ relative to the points z_0, z_1. Under suitable conditions, given z_0, z_1, the law of the mean (1) $d_1f(z_0)=f'(\alpha)$ holds for some α. If α is given, the inverse problem is to show that z_0, z_1 exist. This and related problems have been treated by Pompeiu, Montel and others. The present work extends earlier results. First the following theorem [not all of it new] is proved. Let $f(z)$ be holomorphic in a domain (D). If α is any point of (D), and (C) is an arbitrary closed curve surrounding α and lying interior to (D), then (I) points z_0, z_1 exist on (C) so that (1) holds; (II) to every pair z_0', z_1' in-

terior to or on (C) can be associated a pair z_0, z_1 on (C) such that $d_1f(z_0)=d_1f(z_0')$.

Then properties (I) and (II) are shown to hold for two generalizations: (a) that of the "divided difference" for two functions $f(z)$, $g(z)$; (b) that of divided differences of $f(z)$ of order p (arbitrary positive integer). In case (a), (1) is replaced by

$$\text{(1a)} \qquad \{f(z_1)-f(z_0)\}/\{g(z_1)-g(z_0)\}=f'(\alpha)/g'(\alpha),$$

and in case (b) by

$$\text{(1b)} \qquad d_1d_2\cdots d_pf(z_0)=(1/p!)f^{(p)}(\alpha).$$

Here $p+1$ points $z_0, \cdots, z_p$ are involved in the pth order divided difference $d_1d_2\cdots d_pf(z_0)$. *I. M. Sheffer.*

Miller, Glen T. and Hughes, Howard K. Analytic continuation of functions defined by factorial series. Amer. J. Math. 65, 423–432 (1943). [MF 8700]

If $g(z)$ is holomorphic for $x\geqq-\frac{1}{2}$ and

$$\text{(1)} \qquad |g(z)/g(x)|<M(\epsilon)\exp[\{\pi(k+\tfrac{1}{2})-\epsilon\}|y|]$$

for $x\geqq-\frac{1}{2}$, $|y|>A(\epsilon)$, $\epsilon>0$ arbitrary, $k>0$ integer, (2) for $\zeta\in D$ (bounded region), $x>q(\epsilon)$, $|g(x)/\Gamma(\zeta+x+1)|<N/x^{1+\epsilon}$, then

$$\Omega(z)=\sum_{m=0}^{\infty}g(m)(-1)^{m(k+1)}/z(z+1)\cdots(z+m)$$

$$=\Gamma(z)\int_{-\frac{1}{2}}^{\infty}(\sin k\pi t g(t)/\sin\pi t\Gamma(z+t+1))dt$$

$$+\Gamma(z)[-\tfrac{1}{2}i]^k k\pi\int_{-\infty}^{\infty}(F(t,z)/\cosh\pi t)^{k+1})dt$$

with

$$F(t,z)=\int_0^t\{[\sinh\pi(s-t)]^{k-1}g(-\tfrac{1}{2}+is)/\Gamma(z+\tfrac{1}{2}+is)\}ds.$$

S. Mandelbrojt (Houston, Tex.).

Slobodetzky, L. Sur un théorème de Wigert. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 103–104 (1941). [MF 9622]

An important theorem of Wigert is extended in the following manner. The necessary and sufficient condition that a power series $\sum a_nz^n$ represents an entire function of $z/(1-z)$ of order ρ is that $a_n=\varphi(n)$, where $\varphi(u)$ is an entire function of order $\rho/(\rho+1)$. The proof given by the author yields also Wigert's theorem itself. *G. Szegö.*

Newsom, C. V. The asymptotic behavior of a class of entire functions. Amer. J. Math. 65, 450–454 (1943). [MF 8703]

If s is an integer greater than 1 and ρ_j is a complex constant, the author studies the behavior of the function

$$f(z)=\sum_0^{\infty}g(n)z^{sn},\quad g(n)=1\Big/\prod_1^s\Gamma(n+\rho_j),$$

for large values of z. The function is expressed as a finite sum of powers of z and e^z, with a remainder, and this expression is obtained by using an earlier representation of $f(z)$ due to the author [Amer. J. Math. 60, 561–572 (1938)]. *A. C. Schaeffer* (Stanford University, Calif.).

Ganapathy Iyer, V. The influence of zeros on the magnitude of functions regular in an angle. J. Indian Math. Soc. (N.S.) 7, 1–16 (1943). [MF 9375]

It is well known that the growth of an analytic function $f(z)$ of finite order in an angle is considerably restricted by the possession of zeros at the lattice points in the angle. To prove such results, one studies the function $f(z)/g(z)$, where $g(z)$ is a canonical product with zeros at the lattice points. If one attempts to generalize the set of lattice points in any obvious geometrical way, it is generally difficult to obtain sufficiently detailed information about $g(z)$. Here the author takes as his generalization of the set of lattice points the set of zeros of an entire function $g(z)$ such that $\lim|z|^{-\rho}\log|g(z)|=d$ as $|z|\to\infty$ outside a set of circles, the sum of whose radii is finite. Such a set is said to belong to $L(\rho,d)$. If $f(z)$ is analytic in the angle $|\theta|\leqq\alpha$, we define the maximum modulus $M(r,f)$ and the Phragmén-Lindelöf function $h(\theta,f)$ in the usual way. Sample theorems: if $f(z)$ is of order ρ and finite type, $\alpha=\pi/(2\rho)$, $h(\alpha,f)+h(-\alpha,f)<2d$ and $f(z)$ has at each point of a sequence of $L(\rho,d)$ a zero of order at least as high as the $g(z)$ associated with the sequence, then $f(z)\equiv0$. With the same conditions on $f(z)$, if $\alpha>\pi/(2\rho)$, $h(\pm\alpha,f)<d$, and $\liminf r^{-\rho}\log M(r,f)<d$, then $f(z)\equiv0$. Several examples of sequences of $L(\rho,d)$ are exhibited, and the construction of more is facilitated by the result that, if $\{u_n\}$ belongs to $L(\rho,d)$, so does $\{v_n\}$ if $\sum|u_n-v_n|$ converges. [Reference should be made to related results of B. J. Maitland, Proc. London Math. Soc. (2) 45, 440–457 (1939); these Rev. 1, 49.] *R. P. Boas, Jr.*

Golusin, G. M. Über Koeffizienten der schlichten Funktionen. Rec. Math. [Mat. Sbornik] N.S. 12(54), 40–47 (1943). (Russian. German summary) [MF 8790]

Suppose that $f(z)=z+a_2z^2+\cdots$ belongs to the class S of functions which are regular and schlicht in $|z|<1$ and write $f_1(z)=1+a_2z^2+a_3z^4+\cdots$. The function

$$g(\zeta)=(f((\zeta+z)/(1+\bar{z}\zeta))-f(z))/(f'(z)(1-|z|^2))$$

belongs to S and so $|g(\zeta)|>|\zeta|/4$ $(|\zeta|<1)$. Taking $\zeta=-2z/(1+|z|^2)$ the author obtains the inequality

$$\text{(i)} \qquad |f'(z)|<4|f_1(z)|/(1-|z|), \qquad |z|<1.$$

If $|a_{2n+1}|=O(n^\alpha)$, where $-1<\alpha\leqq1$, then

$$|f_1(z)|=O\{(1-|z|)^{-\alpha-1}\}$$

and (i) gives

$$\text{(ii)} \qquad |f'(z)|=O\{(1-|z|)^{-\alpha-2}\}.$$

If $\alpha>-\frac{1}{2}$, (ii) implies (as is well known) that $|a_n|=O(n^\alpha)$ (all n). Hence, if $\alpha>-\frac{1}{2}$ and $|a_n|=O(n^\alpha)$ is true for odd n, it is true for all n. The author proves two additional inequalities, namely,

$$|f'(z)f'(-z)|\leqq((1+|z|^2)/(1-|z|^2))^2|f_1(z)|^2,$$
$$|f(z)f(-z)|\leqq|z|^2((1+|z|^2)/(1-|z|^2)^2)|f_1(z)|;$$

the first inequality is best possible (equality for $f=z/(1-z^2)$, z real) but the second is not. Finally, let

$$f_p(z)=(f(z^p))^{1/p}=\sum_{n=0}^{\infty}a_n^{(p)}z^{pn+1},\qquad a_0^{(p)}=1.$$

An explicit example is given of a function f_p for which $|a_2^{(p)}|>1$ if $p>2$ [the exact upper bound of $|a_2^{(p)}|$ for $p\geqq2$ has been given by Fekete and Szegö] and the construction also provides a function f for which $|2a_3-a_2^2|>2$ [the maximum of $|2a_3-a_2^2|$ for f in S is $2+4e^{-2}$, but this is not stated]. *D. C. Spencer.*

Golusin, G. M. Zur Theorie der schlichten Funktionen. Rec. Math. [Mat. Sbornik] N.S. 12(54), 48–55 (1943). (Russian. German summary) [MF 8791]

Let G be a simply or multiply connected domain in the S-plane containing $\zeta=\infty$, and let $\mathfrak{F}$ be the family of

functions $F(\zeta)$ which are regular (except for a pole at $\zeta=\infty$) and schlicht in G and (near $\zeta=\infty$) of the form $F(\zeta)=\zeta+\alpha_1/\zeta+\cdots$. For fixed θ and fixed $\zeta_1 \varepsilon G$ the author applies the lemma of M. Schiffer [Proc. London Math. Soc. (2) **44**, 432–449 (1938)] to investigate the maxima of the following three quantities for $F \varepsilon \mathfrak{F}$: (1) $\Re\{e^{2i\theta}\log F'(\zeta_1)\}$, (2) $\Re\{e^{2i\theta}F(\zeta_1)\}$, (3) $\Re\{e^{2i\theta}F''(\zeta_1)/F'(\zeta_1)\}$. Let $z=F_0(\zeta)$ denote an extremal function and let D_0 be the map of G by F_0. In case (1) it is shown that any continuum lying in the complement of D_0 is an arc of a logarithmic spiral (theorem of Grötzsch). Moreover, if G is the exterior of the unit circle $|\zeta|=1$, then the inequality $|\log F'(\zeta)| \leqq -\log(1-1/|\zeta|^2)$ is true throughout $\mathfrak{F}$. In case (2) the complement of D_0 is composed of curves satisfying $\Re(e^{i\theta}(z-F_0(\zeta_1))^{\frac{1}{2}})=\text{const.}$ If G is the exterior of the unit circle, then (in $\mathfrak{F}$)

$$\left|\frac{F(\zeta)}{\zeta}+1-\frac{1}{|\zeta|^2}-2\frac{E(1/|\zeta|)}{K(1/|\zeta|)}\right| \leqq 2-2\frac{E(1/|\zeta|)}{K(1/|\zeta|)}, \quad |\zeta|>1,$$

where E and K are the complete elliptic integrals. In case (3) D_0 is bounded by slits satisfying

$$\Re(e^{i\theta}(F_0'(\zeta_1))^{\frac{1}{2}}/(z-F_0(\zeta_1))^{\frac{1}{2}})=\text{const.},$$

and, if G is the exterior of the unit circle, we have the inequality

$$\left|\frac{\zeta F''(\zeta)}{F'(\zeta)}+\frac{4|\zeta|^2-2}{|\zeta|^2-1}-\frac{4|\zeta|^2}{|\zeta|^2-1}\cdot\frac{E(1/|\zeta|)}{K(1/|\zeta|)}\right| \leqq \frac{4|\zeta|^2}{|\zeta|^2-1}\left(1-\frac{E(1/|\zeta|)}{K(1/|\zeta|)}\right), \quad |\zeta|>1.$$

Finally, let G be a domain containing $\zeta=0$ and consider the family of functions $f(\zeta)$ which are regular and schlicht in G and (near $\zeta=0$) of the form $\zeta+a_2\zeta^2+\cdots$. If $f_0(\zeta)$ is a function maximizing $\Re\{e^{2i\theta}\log(f(\zeta_1)/\zeta_1)\}$ for fixed θ and ζ_1 ($\zeta_1 \varepsilon G$), the boundary of the map of G by f_0 is made up of curves

$$z(t)=f_0(\zeta_1)\cdot 4ce^{-i\theta}t/(1+ce^{-i\theta}t)^2,$$

where c is a constant and t is real. [In the case when G is the interior of $|\zeta|=1$ the author states that f_0 satisfies the equation

$$\left(\frac{f_0'(\zeta)}{f_0(\zeta)}\right)^2\frac{1}{1-f_0(\zeta)/f_0(\zeta_1)}=\frac{\zeta_1(e^{i\theta}\zeta^{-1}\pm e^{-i\theta}\zeta_1^{-1})^2}{(\zeta-\zeta_1)(1-\bar{\zeta}_1\zeta)}.$$

This is false as is seen by letting $\zeta\to 0$. The corrected equation may be written

$$e^{2i\theta}\zeta^2\left(\frac{f_0'(\zeta)}{f_0(\zeta)}\right)^2\frac{1}{1-f_0(\zeta)/f_0(\zeta_1)}=\frac{\zeta}{(\zeta-\zeta_1)(1-\bar{\zeta}_1\zeta)}\left(2|\zeta_1|-\frac{\zeta_1e^{2i\theta}}{\zeta}-\bar{\zeta}_1e^{-2i\theta}\zeta\right).]$$

D. C. Spencer (Stanford University, Calif.).

Ferrand, Jacqueline. Sur un théorème de M. Golusin. C. R. Acad. Sci. Paris **215**, 254–255 (1942). [MF 9494]

The following theorem due to Golusin [Rec. Math. [Mat. Sbornik] N.S. **1(43)**, 273–281 (1936); **2(44)**, 617–619 (1937)] is proved without using Grötsch's principle: if $w=f(z)=z+a_2z^2+\cdots$ is analytic and univalent for $|z|<1$, and if $\rho(\theta)$ denotes the distance from the origin to the first meeting point of $\arg w=\theta$ with Γ, the boundary of the image of $|z|<1$ with respect to $f(z)$, then

$$\int_0^{2\pi}\log\rho d\theta \geqq 0.$$

M. H. Heins (Chicago, Ill.).

Dufresnoy, Jacques. Une propriété des surfaces de recouvrement. C. R. Acad. Sci. Paris **215**, 252–253 (1942). [MF 9493]

Let Σ denote a simply-connected covering surface of the Riemann sphere with area $4\pi S$ and length of contour L. Let there be given q simply-connected mutually disjoint domains D_i (of which certain may reduce to points). With each D_i two integers μ_i and d_i are associated, where (1) $\mu_i>1$; (2) $\sum_{i=1}^{q}(1-1/\mu_i)\geqq 2$; (3) $d_i\geqq 0$, with equality occurring for at least two d_i. Consider the set of μ_i corresponding to $d_i=0$ and let μ denote the second of these arranged in decreasing order. Finally, let

$$\sum_{i=1}^{q}(1-1/\mu_i)\min(d_i, m)<1+m\left[\sum_{i=1}^{q}(1-1/\mu_i)-2\right]$$

for $m\geqq\mu$. Then, if the disks of Σ over D_i are, with the exception of d_i of them, μ_i-sheeted, there exists a number L_0 (depending only on the D_i, μ_i, d_i) such that $L>L_0$ when $S>\frac{1}{2}$. The proof is based on methods related to those of Ahlfors which have been developed in the thesis of Dufresnoy [Ann. Sci. École Norm. Sup. **58**, 179–259 (1941).] This theorem is applied to special classes of functions meromorphic in the finite plane. *M. H. Heins* (Chicago, Ill.).

Dufresnoy, Jacques. Un critère de famille normale. C. R. Acad. Sci. Paris **215**, 294–296 (1942). [MF 9500]

The theorem of the paper reviewed above is applied to deduce a criterion for the normality of a family of meromorphic functions. Bounds are given for the spherical derivative, area of Riemannian image, characteristic function, logarithm of the modulus for functions of the family. *M. H. Heins* (Chicago, Ill.).

***Weyl, Hermann. Meromorphic Functions and Analytic Curves.** Annals of Mathematics Studies, no. 12. Princeton University Press, Princeton, N. J., 1943. ix+269 pp. $3.50.

The subject matter of this tract finds its origin in the investigations on meromorphic curves initiated in 1938 by H. and J. Weyl [Ann. of Math. (2) **39**, 516–538 (1938)]. The theory of meromorphic curves seeks to extend Nevanlinna's theory of meromorphic functions to systems of meromorphic functions having a common domain of definition. This tract gives an account of the theory as its stands at the present time. As might be expected, the theory of meromorphic curves calls for a considerable amount of material from various fields such as projective spaces and the theory of Riemann surfaces. These and related topics are developed to the extent they are needed. Profound generalizations of the Nevanlinna theory are given, such as the extension of the theory to the case where the common domain of definition of the meromorphic functions considered is a given abstract Riemann surface.

An idea of the subject matter may be had from the table of contents. Chapter I is concerned with geometric and function-theoretic foundations (unitary geometry, analytic curves defined on a Riemann surface). Chapter II treats the first main theorem for meromorphic curves. Applications

to exponential curves are given [cf. J. Weyl, Duke Math. J. **10**, 123–143 (1943); these Rev. **4**, 216]. The second main theorem for meromorphic curves is considered in chapter III. Chapter IV extends the theory to the case of analytic curves defined on an abstract Riemann surface. Chapter V treats the defect relations. A bibliography is given. The present investigations are coordinated with the earlier related work of Borel, Bloch and H. Cartan. *M. H. Heins.*

Roure, Henri. Sur une généralisation des fonctions hyperabéliennes d'Émile Picard. C. R. Acad. Sci. Paris **214**, 783–784 (1942). [MF 9451]

Suppose that $G=aUU_0+bVV_0+cWW_0$ and $H=a'U'U_0'+b'V'V_0'+c'W'W_0'$ are two forms with real coefficients, the subscript zero denoting the conjugate complex. Let

$$x\to X=(Ax+By+C)(A''x+B''y+C'')^{-1},$$
$$y\to Y=(A'x+B'y+C')(A''x+B''y+C'')^{-1}$$

and

$$z\to Z=(Dz+Et+F)(D''z+E''t+F'')^{-1},$$
$$t\to T=(D'z+E't+F')(D''z+E''t+F'')^{-1}$$

be the automorphisms of the forms G and H, respectively, admitting only integral coefficients. Call $\mathfrak{G}$, $\mathfrak{H}$ the two groups. In order that $\mathfrak{G}$ and $\mathfrak{H}$ be infinite it is necessary that the coefficients of each form do not all have the same sign. Assume $a, b, a', b'>0$ and $c=-h<0$, $c'=-h'<0$. The group $\mathfrak{G}$ is discontinuous for the inside of the region D defined by $a|x|^2+b|y|^2-h<0$, similarly for $\mathfrak{H}$. Then

$$\sum_{\alpha,\beta}\left[\left|\frac{\partial(X,Y)}{\partial(x,y)}\,\frac{\partial(Z,T)}{\partial(z,t)}\right|\right]^{2m}$$

is convergent for $m>3$. Application of the classical methods of Picard and Poincaré yields a generalization of the usual modular forms and functions. *O. F. G. Schilling.*

Bergman, Stefan. The behavior of the kernel function at boundary points of the second order. Amer. J. Math. **65**, 679–700 (1943). [MF 9395]

Consider a domain D in the space of two complex variables (z_1, z_2). The author gives a geometric classification of the boundary points of D depending upon the behavior of certain analytic surfaces which pass through the point. He considers four cases: (a), (b), (c) and (d). As an example, case (b) holds for a boundary point Q if there exists an analytic surface a segment of which lies in the boundary of D and Q is an interior point of this segment. Having given this geometric classification of boundary points, the author next gives an analytic classification of the boundary points in terms of the behavior of the expression

$$(1)\qquad [|Z_1-z_1|^2+|Z_2-z_2|^2]^{t/2}K_D(z_1, z_2, \bar{z}_1, \bar{z}_2)$$

as the point (z_1, z_2) approaches the boundary point $Q(Z_1, Z_2)$. (Here K is the kernel of domain D. In general, the kernel becomes infinite as (z) approaches the boundary.) If the expression (1) remains between two positive bounds when (z_1, z_2) approaches Q in a suitably restricted manner, then Q is called an R-point of order t. If, furthermore, the expression (1) possesses a definite positive limit when (z_1, z_2) approaches Q in a suitably restricted manner, then Q is called an L-point of order t.

Under certain additional hypotheses, the author asserts that a boundary point possessing one of the properties (a), (b), (c) or (d) is an R-point of order t with $t=3, 2, 4, 0$, respectively. In an earlier paper the author investigated in some detail boundary points of the third order [J. Reine Angew. Math. **169**, 1–42 (1932); **172**, 89–128 (1934)]. In the present paper he investigates points of the second order. The method used is to compare the domain D with certain standard domains S also having Q as a boundary point and either containing D or contained in D. The standard domains considered are such that the kernel for them can be easily calculated or at least estimated. The author proves two main theorems. The first theorem gives sufficient conditions for a boundary point to be an R-point of the second order, while the second theorem gives sufficient conditions for a boundary point to be an L-point of the second order. For the proof of the second theorem, the author constructs suitable interior and exterior domains of comparison for the domain D. A known property of the kernel states that the relation $I\subset D\subset A$ implies

$$K_A(z_1, z_2, \bar{z}_1, \bar{z}_2)\leqq K_D(z_1, z_2, \bar{z}_1, \bar{z}_2)\leqq K_I(z_1, z_2, \bar{z}_1, \bar{z}_2).$$

From this and a knowledge of the behavior of the kernel for the inner comparison domain I and the outer comparison domain A the author obtains the desired result.

W. T. Martin (Syracuse, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Jacob, M. Über eine Anwendung der Laplace-Transformation auf die Summation Fourierscher Reihen und trigonometrischer Interpolationspolynome. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 390–394 (1941). [MF 9601]

The author considers the following transforms of the partial sums of a Fourier series

$$\sigma_n^{(\lambda,\delta)}(x)=a_0/2+\sum_{\nu<\omega+1}(1-\nu^\lambda/(\omega+1)^\lambda)^\delta(a_\nu\cos\nu x+b_\nu\sin\nu x)$$
$$=\int_0^{2\pi}f(t)K_{\omega+1}^{(\lambda,\delta)}(t-x)dt,\qquad \delta>0;\ \lambda=1, 2,$$

and the analogous generalizations of the Bernstein interpolation polynomials

$$\tilde{\sigma}_{n,\omega}^{(\lambda,\delta)}(x)=\int_0^{2\pi}f(t)K_{\omega+1}^{(\lambda,\delta)}(t-x)d\varphi_{2n+1}(t),$$

$\varphi_{2n+1}(t)=x_k^{(n)}$ in $x_k^{(n)}\leqq t<x_{k+1}^{(n)}$, $x_k^{(n)}=2\pi k/(2n+1)$. The kernel $K^{(\lambda,\delta)}$ has been expressed in terms of Bessel functions by G. Doetsch [Compositio Math. **1**, 85–97 (1934)]. The author obtains an expression for $K^{(1,\delta)}$ in terms of W. H. Young's function

$$C_{1+\delta}(t)=\{t^{1+\delta}/\Gamma(1+\delta)\}\int_0^1(1-u)^\delta\cos tu\,du$$

by the use of Laplace transforms. Hence he obtains integral formulas for $\sigma_n^{(\lambda,\delta)}(x)$ [known] and $\tilde{\sigma}_n^{(\lambda,\delta)}(x)$ [new]; these he uses to estimate the order of the remainders $\sigma_n^{(\lambda,\delta)}(x)-f(x)$, $\tilde{\sigma}_{n,n}^{(\lambda,\delta)}(x)-f(x)$ in case $f(x)$ satisfies a Lipschitz condition of order α, $0<\alpha\leqq 1$. The remainders are of order $n^{-\alpha}$ when $\alpha\neq 1$, $\delta>\alpha$, for both σ and $\tilde{\sigma}$; when $\alpha=1$, the remainders are (for $\delta\geqq 1$) of order $n^{-1}\log n$ for $\lambda=1$, but of order n^{-1} for $\lambda=2$; when $\delta=\alpha<1$, the order is $n^{-\delta}\log n$ in all cases [more precise statements are given in the paper]. The case $\lambda=\delta=1$ was considered by Nikolski [Bull. Acad. Sci. URSS **4**, 501–508 (1940); these Rev. **2**, 279]. *R. P. Boas, Jr.* (Cambridge, Mass.).

The reference should be to Rec. Math [Mat. Sbornik] N.S. 12(54), 3-27 (1942); these Rev. 5,5.

Kac, M. On the distribution of values of trigonometric sums with linearly independent frequencies. Amer. J. Math. 65, 609–615 (1943). [MF 9389]

Let

$$f(t)=\sum_1^n C_k \cos 2\pi\lambda_k t$$

be a trigonometric sum, where the C_k are real and the λ_k are linearly independent. Let $N_T(a)$ be the number of roots of the equation $f(t)=a$ lying in the interval $-T\leqq t\leqq T$. The purpose of the present paper is to give an explicit formula for the limit

$$E(a)=\lim_{T\to\infty}(N_T(a)/2T).$$

The formula is given in the form of a double infinite integral whose integrand contains infinite products. [The first problem of this type seems to have been treated by Stein [Proc. Cambridge Philos. Soc. 31, 455–467 (1935)], who dealt only with the case $k=2$.] *R. Salem.*

Fréchet, Maurice. Les fonctions asymptotiquement presque-periodiques continues. C. R. Acad. Sci. Paris 213, 520–522 (1941). [MF 9221]

The author considers functions which can be expressed in the form $f(t)=p(t)+\omega(t)$, where $p(t)$ is Bohr almost periodic (a.p.) and $\omega(t)$ is a continuous function which approaches zero as $t\to\infty$; he calls such functions (defined either on the whole real axis or the half line $t\geqq\alpha$) asymptotically almost periodic (a.a.p.) functions. He states that certain relaxations of the uniformity conditions in the definition of Bohr a.p. or Bochner normal functions lead to this class of functions, and refers to applications of these functions in probability theory which he has made in former papers without (apparently) specifically formulating the definition or studying the properties of a.a.p. functions as such. He now takes up the task of listing these properties, such as behavior under differentiation, integration, substitution in continuous functions, etc. These properties are more or less what one might expect, with the a.p. part playing the major and the corrective part the minor role. The author says he will give their proofs in another paper.

R. H. Cameron (Cambridge, Mass.).

Mitchell, Josephine. On double Sturm-Liouville series. Amer. J. Math. 65, 616–636 (1943). [MF 9390]

The double Sturm-Liouville series of an integrable function $f(x, y)$ is formed from a set of orthonormal Sturm-Liouville functions in the same way as its double Fourier cosine series. The double Sturm-Liouville series is equisummable $(C, 1, 1)$ with the double Fourier cosine series at almost all points (x, y) and in particular at all (x, y) such that

$$\int_0^h\int_0^h |f(x+u, y+v)|\,du\,dv=O(hk)$$

as $h, k\to 0$, a condition which is satisfied almost everywhere if $f(x, y)\log^+|f(x, y)|$ is integrable. The two double series are equiconvergent if $f(x, y)$ is of bounded variation in the sense of Tonelli. The two double series are equisummable by a modified Poisson summation method suggested by S. Bergman [Abstract in Bull. Amer. Math. Soc. 47, 555 (1941)] if $f(x, y)$ is integrable; the corresponding theorem for simple series and functions of one variable is also given.

R. P. Boas, Jr. (Cambridge, Mass.).

Chandrasekharan, K. The absolute summability of series of eigenfunctions. J. Indian Math. Soc. (N.S.) 7, 25–30 (1943). [MF 9377]

Minakshisundaram has studied problems concerning the summability (R, μ, k) of the Fourier series of a function $f(x, y)$ in terms of the eigenfunctions of the boundary value problem $\nabla^2\omega+\mu\omega=0$ in D, $\omega(x, y)=0$ on the boundary of D, where D is a suitably restricted finite domain [J. Indian Math. Soc. (N.S.) 6, 153–167 (1942); these Rev. 4, 246]. The present paper develops a corresponding theory for absolute summability. *F. W. Perkins.*

Bary, N. Sur la stabilité de certaines propriétés des systèmes orthogonaux. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 342–345 (1941). [MF 9567]

This paper surveys without proofs properties shared by orthogonal sets "near" each other in various degrees. The details were made available in a later paper [same C. R. (N.S.) 37, 83–87 (1942); these Rev. 4, 272]. *H. Pollard.*

~~The reference should be to Rec. [Mat. Sbornik]~~

Erdélyi, A. Note on an inversion formula for the Laplace transformation. J. London Math. Soc. 18, 72–77 (1943). [MF 9700]

If $g(s)=\int_0^\infty e^{-st}f(t)\,dt$, $\Re(s)>0$, the inversion formula expresses $f(t)$ in terms of the values of $g(s)$ at a suitable discrete set of points, and a related interpolation formula expresses $g(s)$ in terms of the same set of values. [These differ from the formulas announced by the author in Philos. Mag. (7) 34, 533–537 (1943); these Rev. 5, 4.] Let $\{\lambda_n\}$ be a sequence of distinct numbers of positive real part, such that $\sum\Re(\lambda_n)/(1+|\lambda_n|^2)$ diverges, so that $\{e^{-\lambda_n t}\}$ is closed $L^2(0, \infty)$. From the sequence $\{e^{-\lambda_n t}\}$ one can obtain an orthonormal set $\{\varphi_n\}$, $\varphi_n(t)=\sum_{m=0}^n c_{mn}e^{-\lambda_m t}$, where the c_{mn} can be explicitly exhibited in terms of the λ_n. Then formally $f(t)=\sum_{n=0}^\infty\varphi_n(t)\sum_{m=0}^n c_{mn}g(\bar\lambda_m)$, as one sees by expanding $f(t)$ in terms of the φ_n and identifying the coefficients in terms of g. If the expansion is substituted into the integral defining $g(s)$ and integrated term by term, one obtains an interpolation formula expressing $g(s)$ in terms of $g(\bar\lambda_n)$. A rigorous L^2 theory of these formulas is presented.

R. P. Boas, Jr. (Cambridge, Mass.).

Kober, H. A note on Hilbert transforms. J. London Math. Soc. 18, 66–71 (1943). [MF 9699]

The Hilbert transform $\mathfrak{H}F$ of $F(t)$ is defined as $\pi^{-1}\int_{-\infty}^\infty F(t)(t-x)^{-1}dt$, the integral being a principal value. Let K be the class of functions F such that both F and $\mathfrak{H}F$ belong to $L(-\infty, \infty)$; this class has been studied previously by the author [Bull. Amer. Math. Soc. 48, 421–426 (1942); these Rev. 4, 40]. Generalizing a result of Titchmarsh [Introduction to the Theory of Fourier Integrals, Oxford, 1937, p. 144] the author proves that, if $F\in L(-\infty, \infty)$, $0<\lambda<1$, and $\mu>\frac{1}{2}-\frac{1}{4}\lambda$, then

$$\left\{\int_{-\infty}^\infty\frac{|HF|^\lambda}{(1+x^2)^\mu}dx\right\}^{1/\lambda}\leqq A_{\lambda,\mu}\int_{-\infty}^\infty|F(t)|\,dt;$$

no smaller value of μ will make the left side always finite. The proof goes via the function $f(e^{i\theta})=(1+e^{i\theta})^{-2}F(\tan\frac{1}{2}\theta)$, by the use of which properties of the conjugate of a periodic function can be carried over to the transform $\mathfrak{H}$, and which leads to various sufficient conditions for $F\in K$. Inequalities are given for the related transform

$$\mathfrak{K}F=\frac{1}{\pi}\int_{-\infty}^\infty\left\{\frac{1}{t-x}-\frac{t}{t^2+1}\right\}F(t)dt,$$

where $F(t)$ satisfies less restricted conditions than belonging to L. Applications include the following result on Fourier transforms: if $F(t)$ is measurable,

$$F(t)\log^+|(1+t^2)F(t)|\in L(-\infty,\infty),$$

$G(x)$ is the Fourier transform of $F(t)$, and $G(a)=G(b)=0$, then the function $\varphi(x)$ which is $G(x)$ in (a, b) and 0 outside is itself a Fourier transform. *R. P. Boas, Jr.*

Mohan, B. A class of kernels. J. Benares Hindu Univ. Silver Jubilee Number, 134–137 (1942). [MF 9678]

A collection of kernels $P(x)$ such that, if $f(x)$ is self-reciprocal for the Hankel transform of one order, then $\int_0^\infty P(xy)f(y)dy$ is self-reciprocal for the Hankel transform of another order. [For references see these Rev. 3, 234.]

R. P. Boas, Jr. (Cambridge, Mass.).

Differential Equations

***Courant, R. und Hilbert, D. Methoden der Mathematischen Physik. Vols. I, II.** Interscience Publishers, Inc., N. Y., 1943. xiv+469 pp. and xiv+549 pp. $14.00.

Photo-lithoprint reproduction of the second (latest) edition of the first volume (1931) and of the only edition of the second volume (1937). The original publisher was Springer, Berlin.

Abelé, Jean. Système d'entretien à amplitude autostabilisée. C. R. Acad. Sci. Paris **214**, 841–842 (1942). [MF 9458]

The author considers under what conditions the equations

$$x\,dx+y\,dy+2Ry\,dx=0,\quad dx=y\,dt$$

have a stable sinusoidal solution; R is a function of x and y.

N. Levinson (Cambridge, Mass.).

Parodi, Hippolyte. Système simple d'équations analogue à une équation de relaxation unique. C. R. Acad. Sci. Paris **215**, 125–127 (1942). [MF 9481]

The author makes the obvious observation that, if f is given different constant values in each of several regions of the phase plane, the equation $\ddot{s}+f+s=0$ will describe a relaxation oscillation. *N. Levinson.*

Parodi, Hippolyte. Traction électrique automatique et équations de relaxation. C. R. Acad. Sci. Paris **215**, 169–171 (1942). [MF 9483]

The author shows that under certain conditions electrical traction can be described by a relaxation type of equation.

N. Levinson (Cambridge, Mass.).

Parodi, Hippolyte. Détermination graphique du cycle limite solution des équations de relaxation. C. R. Acad. Sci. Paris **215**, 196–199 (1942). [MF 9488]

The author observes that a well-known method of graphical solution of the general first order differential equation can be applied to the equation

$$v\,dv/ds+v\,f(s)+s=0.$$

N. Levinson (Cambridge, Mass.).

Parodi, Hippolyte et Parodi, Maurice. Méthode d'intégration par arcs successifs des équations de relaxation. C. R. Acad. Sci. Paris **215**, 268–270 (1942). [MF 9496]

The author considers the approximate solution of

$$v\,dv/ds+v\,f(s)+s=0$$

by approximating to $f(s)$ by a step function. In each interval where $f(s)$ is constant the equation can be solved explicitly.

N. Levinson (Cambridge, Mass.).

Lusternik, L. Sur un problème limite dans la théorie des équations différentielles non linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 5–8 (1941). [MF 9616]

The extension of results demonstrated for a system of differential equations to another system differing from the original only by terms involving a small parameter is a well-known procedure. Certain nonrigorous attempts were made by Poincaré to extend these results to cases where the systems differed by more than small terms. In this paper Lusternik gives a rigorous extension procedure for the case of systems differing by more than small terms. He uses the notion of locally linear spaces of $(\infty-n)$ dimension [same C. R. (N.S.) **27**, 771–777 (1940); these Rev. **2**, 179]. The author demonstrates his method by considering the equation

$$y''+\phi(x, y, y')+\lambda y=0.$$

If $\phi\equiv 0$, we know that there exists a λ for which the equation has a solution $y(x)$, with $y(a)=y(b)=0$ and $\int_a^b y^2dx=1$, and which vanishes exactly $(k-1)$ times in the interior of (a, b). The author shows how this result can be extended to any ϕ satisfying certain very general conditions. For any such ϕ he proves the existence of a λ for which the equation has at least one solution $y(x)$ of the type just described.

N. Levinson (Cambridge, Mass.).

Pfeiffer, G. Un critère simplifié d'existence de facteurs linéaires dans une forme symbolique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 227–228 (1943). [MF 9245]

A criterium is presented for determining whether a symbolic form of degree n in $n+k$ variables

$$\sum_{\alpha_1,\cdots,\alpha_n}^{n+k} A_{\alpha_1\cdots\alpha_n}dx_{\alpha_1}\cdots dx_{\alpha_n}$$

contains as a factor a linear differential form. Where previous methods lead to a set of dependent relations, the method indicated in this paper leads to independent relations only. *D. J. Struik* (Cambridge, Mass.).

Pfeiffer, G. On the practical application of the criteria of existence of linear factors in symbolic forms. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 289–291 (1943). [MF 9578]

Minakshisundaram, S. On expansion in eigenfunctions of boundary value problems. IV. J. Indian Math. Soc. (N.S.) **7**, 17–24 (1943). [MF 9376]

This paper makes use of material from paper III of the series [J. Indian Math. Soc. (N.S.) **6**, 153–167 (1942); these Rev. **4**, 246]. Let D be a finite domain of the x, y plane, simply or multiply connected, whose boundary C consists of a finite number of regular arcs. Let $\mu_1<\mu_2<\mu_3<\cdots$ be the positive eigenvalues of the boundary value problem $\nabla^2\omega+\mu\omega=0$ in D, $\omega(x, y)=0$ on C, and let $\omega_n(x, y)$, $n=1, 2, 3, \cdots$, be the corresponding eigenfunctions. Also

let $\sum a_n$ be a series of constants, and set

$$R_k(w; x, y) = R_k(w) = \sum_{\mu_n \leq w} (1-\mu_n/w)^k a_n \omega_n.$$

The present paper deals with the problem of determining necessary and sufficient conditions that a given series of the form $\sum a_n\omega_n$ should be the Fourier series of a function $f(x, y)$ integrable in D, that is, that there should exist a function $f(x, y)$ such that $a_n = \iint_D f\omega_n dxdy$, $n=1, 2, 3, \cdots$. Conditions are given for various function spaces. As an example of the type of result obtained the following theorem may be stated. Let L^q, $q \geqq 1$, be the space of measurable functions, the qth powers of whose moduli are integrable in D, with

$$\|f\| = \left\{ \iint_D |f(x, y)|^q dxdy \right\}^{1/q}.$$

A necessary and sufficient condition that $\sum a_n\omega_n$ be the Fourier series of a function $f(x, y) \in L^q$, $1 \leqq q < \infty$, is that

$$\lim_{w, w' \to \infty} \|R(w; x, y) - R(w'; x, y)\| = 0.$$

F. W. Perkins (Hanover, N. H.).

Weinstein, Alexander and Chien, Wei Zang. On the vibrations of a clamped plate under tension. Quart. Appl. Math. 1, 61–68 (1943). [MF 8188]

"The object of the present paper is the computation of the fundamental frequency of a vibrating clamped plate under uniform tension." Actually the paper goes farther by including higher frequencies and other shapes of a plate. On the basis of previous more theoretical publications on variational methods by Weinstein, the numerical calculation, in particular of rather accurate lower bounds, is given.

R. Courant (New York, N. Y.).

Gevrey, Maurice. Sur le problème de la dérivée oblique relatif aux équations linéaires aux dérivées partielles ou intégrodifférentielles du type elliptique canonique à deux variables. C. R. Acad. Sci. Paris 213, 635–637 (1941). [MF 9645]

See these Rev. 4, 143 for a statement of the contents of this paper. *F. G. Dressel* (Durham, N. C.).

Gevrey, Maurice. Sur les problèmes aux limites comportant une dérivée oblique et concernant le type elliptique à m variables. C. R. Acad. Sci. Paris 214, 854–855 (1942). [MF 9460]

The paper states methods for solving the mixed boundary value problem for the author's generalization of the elliptic operator

$$\sum a_{ij}(\partial^2/\partial x_i \partial x_j), \quad i, j = 1, \cdots, m; \; m > 2.$$

The results are the extension from the plane region $m=2$ [cf. same C. R. 214, 206–208 (1942); these Rev. 4, 143] to the case $m>2$. *F. G. Dressel* (Durham, N. C.).

Dive, Pierre. Ondes ellipsoïdales autour d'une source en mouvement uniforme. C. R. Acad. Sci. Paris 215, 273–275 (1942). [MF 9498]

The problem of determining the form of the wave fronts due to a moving source is considered. On the basis of two assumptions it is shown that the wave fronts will be of ellipsoidal form. *M. Wyman* (Edmonton, Alta.).

Olevsky, M. Solution du problème de Cauchy et de certains problèmes limites pour l'équation des ondes, l'équation de la chaleur et l'équation de Laplace dans les espaces à courbure constante. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 282–287 (1941). [MF 9571]

Let $ds^2 = \sum g_{ij} dx^i dx^j$ $(i, j = 1, \cdots, n; n \geqq 3)$ be the linear element of a space of constant curvature k and $\Delta_2 u$ be the second differential parameter of Beltrami. The paper gives the explicit solutions for the two Cauchy problems: (1) $\Delta_2 u = u_{tt}$, $u(P, 0) = f(P)$, $u_t(P, 0) = \varphi(P)$, where $P = (x_1, \cdots, x_n)$; (2) $\Delta_2 u = u_t$, $u(P, 0) = f(P)$. To obtain the results the author defines the mean value $V(u, P, s)$ of a function u to be

$$V(u, P, s) = (1/w_n) \int_{F_s} u dw, \qquad w_n = \int_{F_s} dw,$$

where F_s is the surface of a "sphere" with center at P and geodesic radius s. He states that V satisfies the equation

$$\partial^2 V/\partial s^2 + (n-1)\sqrt{k} \cot(\sqrt{k}s) \partial V/\partial s = \int_{F_s} \Delta_2 u dw.$$

The Cauchy problems (1) and (2) are changed into the corresponding problems for the function V. The function V can be determined by the method of separation of variables. Finally, u is found by using $\lim_{s \to 0} V = u$. The paper also gives the explicit solution for the first boundary problem of the elliptic equation $\Delta_2 u = 0$ in case the boundary is a "sphere." *F. G. Dressel* (Durham, N. C.).

Kostitzin, Vladimir A. Sur l'équation de la chaleur dans le cas d'une sphère soumise à des conditions spéciales. C. R. Acad. Sci. Paris 213, 972–974 (1941). [MF 9663]

Write the heat equation

$$(*) \qquad \nabla^2 u = m^2 \, \partial u/\partial t,$$

m a constant, in spherical coordinates (r, ϕ, ψ). The author gives a formal solution of (*) in a sphere of radius R and satisfying the conditions

$$u(r, \phi, \psi, 0) = w(r, \phi, \psi),$$

$$Eu(R, \phi, \psi, t) = r \frac{\partial u}{\partial r}\bigg|^{r=R} + \Psi(\phi, \psi, t).$$

The infinite series solutions are obtained by the method of separation of variables. [Another paper by the same author precedes this one [same C. R. 214, 47–49 (1942); cf. these Rev. 4, 145].] *F. G. Dressel* (Durham, N. C.).

Piscounov, N. Intégration des équations de la théorie des couches frontières. Bull. Acad. Sci. URSS Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 35–48 (1943). (Russian. French summary) [MF 8527]

Under the assumption (1) $\phi(0) = \theta(0)$, $\lim_{y \to \infty} \phi(y) = 0$, $\theta(0) > \phi(y)$ for $y > 0$, $\phi(y) \leqq \theta(0) - ky$ for $y < \epsilon_1$, $\epsilon_1 > 0$ arbitrarily small, the author proves the existence of a solution of von Mises' boundary layer equation (2) $u_{yy} = (\theta(x) - u)u_x$ under the conditions $u(0, y) = \phi(y)$, $u(x, 0) = \theta(x)$, $\lim_{y \to \infty} u(x, y) = 0$, where θ is continuous and nondecreasing. He uses the following theorem. Let u be a solution of (2) defined in $A\{x \leqq X; -R \leqq y \leqq R\}$. There exist upper bounds for $|\partial^k u/\partial y^k|$ which depend only upon $\max |u(x, y)|$, $(x, y) \in A$. The proof of this theorem involves certain evaluations for solutions of equations of parabolic type due to S. Bernstein [C. R. (Doklady) Acad. Sci. URSS (N.S.) 18, 385–389 (1938)]. If, in addition to (1), ϕ satisfies the con-

ditions $\phi''(y) \geqq 0$ for $y \leqq \epsilon$, $\epsilon > 0$ arbitrarily small, then the solution is unique. The method of attack, which is an important step in the development of the idea of von Mises in reducing the boundary layer equation to the form (2) and approximating (2) by differential-difference equations, is of considerable mathematical interest. The author uses also some results of E. Rothe [Math. Ann. **102**, 650–670 (1930)]. *S. Bergman* (Providence, R. I.).

Functional Analysis, Ergodic Theory

Schatten, Robert. On reflexive norms for the direct product of Banach spaces. Trans. Amer. Math. Soc. **54**, 498–506 (1943). [MF 9522]

In a previous paper the author has defined the "direct product" $E_1 \otimes E_2$ of two arbitrary Banach spaces [cf. Trans. Amer. Math. Soc. **53**, 195–217 (1943); these Rev. **4**, 161]. A norm for the expressions $f_1 \otimes g_1 + \cdots + f_n \otimes g_n$ induces a norm N', termed the associate norm, for the expressions from the conjugate spaces $\bar{E}_1$ and $\bar{E}_2$. The author considers the relationship of the associate norm with the norm of the conjugate space of $E_1 \otimes E_2$ and also the question of when the associate of a cross norm is itself a cross norm. A norm N is reflexive if $N'' = N$; N is minimal if it is the least norm of all those with the same associate. The author shows that $N' = N'''$, that reflexiveness and minimality are equivalent and that the associate of a uniformly convex norm is the norm for the conjugate space if and only if N is reflexive. If E_1 and E_2 are reflexive then the associate of every cross norm is also a cross norm. The author also discusses in the present paper those expressions for which the rank is bounded, that is, those for which one has essentially only a fixed finite number of terms. One considers these either for $E_1 \otimes E_2$ or $\bar{E}_1 \otimes \bar{E}_2$. It is shown that the values of a cross norm for such a set of expressions do not determine it. *F. J. Murray.*

Mackey, George W. On convex topological linear spaces. Proc. Nat. Acad. Sci. U. S. A. **29**, 315–319 (1943), and Erratum **30**, 24 (1944). [MF 9445]

The author studies convex linear topological space X with the aid of various concepts centering around the notion of a linear system as developed previously [Proc. Nat. Acad. Sci. U. S. A. **29**, 216–221 (1943); these Rev. **5**, 38]. A linear manifold L of linear functionals defined on X is said to be associated with a particular convex topology in X if L consists of those functionals which are continuous in the convex topology of X. It is asserted that, if L is an arbitrary total manifold, there exists a convex topology of X associated with L, and, indeed, both a weakest and a strongest topology of this character. The principal tools used in establishing this and other results are the concepts of a pseudo-norm and a pseudo-norm set [defined in the author's paper referred to above]. If the topology of X is, in the above sense, weakest or strongest, X is called relatively weak or relatively strong, respectively. Sample theorems: X is metrizable if and only if it is relatively strong and L is the union of an ascending sequence of pseudo-norm sets. If X is relatively weak, it is normable if and only if it is finite dimensional and it is metrizable if it is isomorphic to the space (s) of Banach. The paper also contains theorems about four types of completeness, and about spaces of the second category. [In lines 22–23 of the first page the phrase "whose continuous linear functionals" should be replaced by "the pseudo-norm sets of whose continuous pseudo norms."] *A. E. Taylor* (Los Angeles, Calif.).

Fan, Ky. Sur quelques notions fondamentales de l'analyse générale. J. Math. Pures Appl. (9) **21**, 289–368 (1942). [MF 9306]

This paper is divided into four chapters which deal with rather diverse topics in the theory of abstract spaces. Many of the concepts treated were first introduced by Fréchet. Chapter I studies abstract polynomials and the representation of continuous functions as limits of sequences of polynomials. The spaces used are various forms of vector spaces. If $F(x)$ is a function of a variable x in a metric affine space [Fréchet, Espaces Abstraits, Gauthier-Villars, Paris, 1928, p. 144] with values in a second such space then $F(x)$ is called an abstract polynomial of order n if its differences of order $(n+1)$, for arbitrary increments, vanish identically and the nth differences do not vanish identically. A special type of space, said to be of category T', is introduced and for these the following theorem is proved. Let E_1, E_2 be two spaces of category T', and C a compact and closed subset of E_1. Every continuous function $F(x)$ defined on C with values in E_2 can be represented as a limit of a sequence of polynomials which are defined in all of E_1 and the convergence is uniform on C. Although the definition of T' is rather complicated, the author points out that many well-known function spaces are subsumed under this general classification.

Chapter 2 studies the differential of an abstract continuous function. The definition used is a generalization of one first given by Hadamard [Scripta Univ. Ab. Bib. Hierosolymitanarum, Jerusalem, 1923] for classical analysis and later generalized by Fréchet [J. Math. Pures Appl. (9) **16**, 233–250 (1937)]. For certain types of spaces the differential is shown to possess many of the properties of the ordinary differential. This is shown by the consideration of such topics as constant functions, functions of functions, change of variables and so on.

The third chapter considers the notion of dimension first introduced by Fréchet [see Espaces Abstraits, p. 30]. Two types of dimension are distinguished, and several properties of them are proved. The fourth chapter is concerned with the topological characterization of an arc and of a ray. [See also a paper by the author, C. R. Acad. Sci. Paris **213**, 518–520 (1941); these Rev. 5, 103.] In order that a set in a space of F. Riesz be homeomorphic to a closed interval of the real line it is necessary and sufficient that it be separable, locally connected and irreducibly connected between two of its points. According to the author the novelty of the characterization lies not only in the generality of the spaces but also in the fact that no compactness is used. A characterization of a topological ray is also given. *D. Montgomery* (Princeton, N. J.).

Fan, Ky. Les fonctions asymptotiquement presque-périodiques d'une variable entière et leur application à l'étude de l'itération des transformations continues. Math. Z. **48**, 685–711 (1943). [MF 9008]

Let S be a metric space and $f_1, f_2, \cdots$ a sequence of its points which satisfies the following conditions: to each $\epsilon > 0$ correspond positive integers l, N such that among any l consecutive positive integers there is at least one integer τ for which $(f_n, f_{n+\tau}) < \epsilon$ when $n \geqq N$. The author calls such a sequence $\{f_n\}$ an asymptotically almost periodic (a.a.p.)

– Please see the review of another paper by the author, these Rev. 8, 519.

function of the integer variable n and gives other equivalent definitions, all based on Fréchet's definition of an a.a.p. function of a continuous variable [C. R. Acad. Sci. Paris 213, 520–522 (1941), these Rev. 5, 96]. In particular, if S is a Banach space, $\{f_n\}$ is an a.a.p. function of n if and only if it is the sum of an almost periodic sequence and a sequence which approaches zero as $n\to\infty$

The author applies these sequences to a transformation T whose iterates T, T^2, T^3, $\cdots$ are equally continuous and which takes a closed compact set E in S into a part of E; he shows that $T^n x$ is an a.a.p. function of n for each x in E. Further, if $f(x)$ is continuous on E and has its values in a Banach space E_1,

$$\mu(f,x)=\lim_{n\to\infty}\frac{f(Tx)+\cdots+f(T^nx)}{n}$$

is defined and uniformly continuous on E and satisfies $\mu(f,x)=\mu(f,Tx)$. A condition of transitivity is given, and other theorems on transformations are proved, including applications to probability theory. *R. H. Cameron.*

Lorch, Edgar R. The theory of analytic functions in normed Abelian vector rings. Trans. Amer. Math. Soc. 54, 414–425 (1943). [MF 9519]

The author is concerned with the problem of establishing the classic theory of analytic functions over as broad a domain as possible. He takes for his domain a normed Abelian vector ring, that is, a commutative ring R with unit, which is also a normal linear complex vector space such that $|a\cdot b|\leqq|a|\cdot|b|$. It is remarked that, if R is a field or if $|a\cdot b|=|a|\cdot|b|$, then it is the set of complex numbers. The mapping $T_a x=ax$, with a fixed in R, is called the regular representation of R and $a\to T_a$ is an algebraic and topological isomorphism. By this remark the theory of linear transformations is applicable to R [cf. Trans. Amer. Math. Soc. 52, 238–248 (1942); these Rev. 4, 247] and one may speak of the spectrum of an element a in R. The ring R is reducible if R is the sum of two disjoint subrings R_1, R_2 distinct from R. It is then shown that R is irreducible if and only if the spectrum of each point of R is a connected set. It is moreover proved that $\exp(z)\equiv\sum z^n/n!$ converges for all z in R and $\exp(z)$ is simply periodic if and only if R is irreducible. A brief discussion of the Cauchy theory of functions is given, and the theory of ideals, insofar as applicable to the author's work, is covered. The set of regular elements of R forms a topological group relative to multiplication, which is open and hence is expressible as $\sum G_\alpha$, the sum of disjoint maximal open connected sets. The set containing the unit e of the ring is called G_1, and it is shown that it consists precisely of the elements a of the form $\exp(b)$ with b in R. It is also shown that, if there is a set $G_\alpha\neq G_1$, there are infinitely many. The paper closes with a discussion of the separation of the ring R by rectifiable curves. Notions of singular, exterior and interior elements are defined relative to a simply closed rectifiable curve C, and for the case in which R is irreducible it is shown that R is partitioned into such points by C. If the function $z(t)$ defining C has a derivative $z'(t)$ at some t and if $z'(t)$ is a regular element, then C has interior points.

H. H. Goldstine (Aberdeen, Md.).

Murray, F. J. The solution of linear operator equations. J. Math. Phys. Mass. Inst. Tech. 22, 148–157 (1943). [MF 9255]

The details of the solution for f of the equation $Tf=g$, T being a linear (bounded or closed) operator in Hilbert space, are given. The treatment is illustrated with an example showing that

$$Tf=\int_0^{2\pi}\exp(\lambda^2xy)f(x)dx=\lambda^1$$

has no solution. *E. R. Lorch* (New York, N. Y.).

Julia, Gaston. Sur une décomposition canonique des opérateurs linéaires bornés de l'espace hilbertien et sur leur classification. C. R. Acad. Sci. Paris 213, 5–9 (1941). [MF 9148]

The author indicates a classification of all bounded and linear operators A on a Hilbert space H into four classes and obtains certain canonical forms for these classes. In the first class are all A such that both A and A^* establish one-one maps of H onto itself. The operators in the second class have H as their contra-domain but vanish at points other than the origin. If V is the linear closed manifold on which an A in the second class vanishes, then A^* vanishes only at the origin and has $H-V$ as its set of functional values. The third class is characterized by having A vanish only at the origin but having $H-V$, where V is linear and closed, as its contra-domain. In this case A^* vanishes on V. The author puts all other bounded linear operators into the fourth class, which he divides into four subclasses depending upon the zeros of A and A^*. He then decomposes every A into a product of an operator in class I or IV by one or more canonical operators of classes II or III by one or more unitary operators (of class I). Finally, he states that Hermitian operators are of class I or IV.

H. H. Goldstine (Aberdeen, Md.).

Julia, Gaston. Sur la dualité dans l'espace hilbertien. C. R. Acad. Sci. Paris 213, 297–300 (1941). [MF 9161]

Julia, Gaston. Sur la dualité dans l'espace hilbertien et sur le domaine des valeurs des opérateurs bornés de 4e classe. C. R. Acad. Sci. Paris 213, 465–469 (1941). [MF 9169]

Let e_n be a complete orthonormal set in a Hilbert space H and let A be a linear and bounded operator in H. The author considers the vectors $A_n=Ae_n$ and $Y=AX=\sum x_nA_n$, where $X=\sum x_ne_n$, and proposes the problem of characterizing those vectors Y for which $Y=\sum x_nA_n$ and for which the series converges strongly. He solves the problem for the case of A in classes I, II or III [cf. the preceding review] by considering the equivalent problem of determining a sequence Z_n of vectors such that $(A_m,Z_n)=\delta_{mn}$ and inquiring whether $\sum|(Z_n,Y)|^2$ converges and what value is given by $\sum(Z_n,Y)A_n$. For operators A of class I the problem is quite simple since $Z_n=(A^{-1})^*e_n$ is effective but for A in class II or III a somewhat more complicated analysis is needed.

In the second paper the author considers the case when A is of the fourth class. Here the vectors Z_n exist if and only if each of the e_n is in the contra-domain of A^{**}; and $Y=\sum(Z_n,Y)A_n$ for every Y in the contra-domain of A.

H. H. Goldstine (Aberdeen, Md.).

von Neumann, John. On some algebraical properties of operator rings. Ann. of Math. (2) 44, 709–715 (1943). [MF 9407]

Rings $\boldsymbol{M}$ of operators in Hilbert space are considered here. A property is purely algebraic if it can be expressed within $\boldsymbol{M}$ in terms of the algebraic operations, that is, αA, $A+B$, AB, A^* and 1 (the identity). The author establishes the algebraic character of the (α) definiteness of A (essentially

$A=BB^*$), (β) bound of A (defined in terms of definiteness), (γ) strong convergence of A_n to A and (δ) weak convergence of A_n to A. The proof of (γ) is based on the existence of subsequences of A_n which are Σ-sequences; the definition of Σ-sequences brands them as purely algebraic. The proof of (δ) depends on (γ) and on the fact that weak convergence may be characterized by means of strong convergence.

E. R. Lorch (New York, N. Y.).

Murray, F. J. and von Neumann, J. On rings of operators. IV. Ann. of Math. (2) **44**, 716–808 (1943). [MF 9417]

[Parts I and II were by the same authors and appeared in Ann. of Math. (2) **37**, 116–229 (1936) and Trans. Amer. Math. Soc. **41**, 208–248 (1937); Part III, by J. von Neumann, appeared in Ann. of Math. (2) **41**, 94–161 (1940); cf. these Rev. **1**, 146.] The authors continue their study of rings of operators. The present paper deals almost exclusively with factors belonging to the cases (I) and (II). The results in the case (I), largely known, form a useful pattern in guiding the reader to a better view of case (II). A principal object is the study of the notion of isomorphism. An algebraic ring isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$ is a 1-1 correspondence leaving 1, αA, A^*, $A+B$ and AB invariant. A spatial isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$ is an isometric mapping of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ (the spaces over which $\mathbf{M}_1$ and $\mathbf{M}_2$ are defined) which transforms $\mathbf{M}_1$ into $\mathbf{M}_2$. The paper considers two questions. (1) When are $\mathbf{M}_1$ and $\mathbf{M}_2$ algebraically isomorphic? (2) When does algebraic imply spatial isomorphism?

It is essential in this study to determine purely algebraic properties, properties independent of the scaffold space $\mathfrak{H}$. A fundamental conclusion based on previous results of von Neumann [cf. the preceding review] is that eight types of closure (some algebraic, some spatial in character) are equivalent. This settles a problem of topology in a happy manner; the notion of subring is purely algebraic. If $\mathbf{M}$ is a ring, $\overline{\mathbf{M}}$ represents the set of rings algebraically isomorphic to $\mathbf{M}$. Fundamental to the paper is the concept $\overline{\mathbf{M}}^p$, where p assumes certain values depending on $\overline{\mathbf{M}}$, $0<p\leqq\infty$. The development begins with integral p. Consider a matrix of operators $\langle A_{ts}\rangle$, $t, s=1, 2, \cdots, p$, $A_{ts}\epsilon\mathbf{M}$; the totality of these matrices forms $\mathbf{M}^p$. Having considered integral p, certain rational and ultimately (for the case (II_1)) real values are introduced; these behave like numerical exponents. The values θ for which $\overline{\mathbf{M}}^\theta=\overline{\mathbf{M}}$ form a multiplicative group $\mathfrak{G}(\overline{\mathbf{M}})$, the fundamental group of $\overline{\mathbf{M}}$. Factors $\overline{\mathbf{M}}$ and $\overline{\mathbf{N}}$ are called commensurable if $\overline{\mathbf{M}}=\overline{\mathbf{N}}^\theta$ or $\overline{\mathbf{N}}=\overline{\mathbf{M}}^\theta$ with $0<\theta\leqq\infty$. As this is an equivalence, commensurable $\overline{\mathbf{M}}$ are thereby united into a genus $\overline{\overline{\mathbf{M}}}$. All rings which are spatially isomorphic form a spatial type $\ddot{\mathbf{M}}$. With these notions problem (2) is answered: for factors in the cases (I) or (II), $\ddot{\mathbf{M}}$ is completely determined by $\overline{\mathbf{M}}$, $\overline{\mathbf{M}}'$ and a certain dimension ratio which cannot here be defined.

An important contribution in this paper is the concept of approximately (app.) finite ring. Four definitions are given whose equivalence is subsequently established. Probably the most suggestive is this. Let $p_1, p_2, \cdots\to\infty$ be integers with p_n a divisor of p_{n+1}. Let $\mathbf{N}_n$ be rings in the case (I_{p_n}), $\mathbf{N}_1\subset\mathbf{N}_2\subset\cdots$. Then $\mathbf{M}=\mathbf{R}(\mathbf{N}_n; n=1, 2, \cdots)$ is app. finite. It is shown that any $\mathbf{M}$ in case (II_1) contains an app. finite subring which is also in case (II_1). Also, all app. finite rings are algebraically isomorphic. Finally, the fundamental group of app. finite rings exhausts the positive numbers. Various examples previously introduced are discussed from the point of view of app. finiteness. Most interesting, however, is the construction of new examples. Complete details cannot be given but a sketch may suffice. If $\mathfrak{G}$ is a countably infinite group, a Hilbert space $\mathfrak{H}$ is constructed using the elements of $\mathfrak{G}$ as indices. The operations $a\to aa_0$, $a\to a_0^{-1}a$, $a\to a^{-1}$ in $\mathfrak{G}$ define in $\mathfrak{H}$ unitary operations U_{a_0}, V_{a_0}, W with $V_{a_0}=WU_{a_0}W$. If $\mathbf{I}$ and $\mathbf{J}$ represent the set of all U_{a_0} and V_{a_0}, respectively, $\mathbf{R}(\mathbf{I})=\mathbf{J}'$, $\mathbf{R}(\mathbf{J})=\mathbf{I}'$. These are factors if the class $\{c^{-1}ac; c\epsilon\mathfrak{G}\}$ of conjugates of a is infinite for every $a\neq 1$. Finally they are approximately finite if $\mathfrak{G}$ is the set sum of finite groups $\mathfrak{G}_1\subseteqq\mathfrak{G}_2\subseteqq\cdots$. This example shows how close these studies are to the theory of group representation.

A new invariant Γ of approximately finite rings is introduced. Using a group background, a factor $\mathbf{M}$ devoid of property Γ is constructed. Thus not all factors in case (II_1) are algebraically isomorphic. An example is given of two factors (II_1) which are not isomorphic and yet each is isomorphic to a subset of the other. This is the present state of knowledge on question (1).

The paper is very carefully cross-indexed; all necessary details are introduced. Thus the results of this series are readily accessible to a moderately initiated reader.

E. R. Lorch (New York, N. Y.).

Fomin, S. Finite invariant measures in the flows. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 99–108 (1943). (Russian. English summary) [MF 8796]

The author generalizes several of the results of Kryloff and Bougoliouboff [Ann. of Math. (2) **38**, 65–113 (1937)] to the case of noncompact phase spaces. The structure of all invariant measures is analyzed under the assumption that the flow is such that it admits at least one finite invariant measure. The necessary and sufficient conditions for that assumption have been established earlier by Oxtoby and Ulam [Ann. of Math. (2) **40**, 560–566 (1939); these Rev. **1**, 18]. In the present paper these conditions are established by a different method, using the space of all uniformly bounded real valued continuous functions over the given phase space with a topology under which it becomes separable. The author shows the existence of transitive invariant measures, and also that an arbitrary invariant measure can be obtained by integration over transitive measures. The paper indicates how certain recent results of Bebutov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 482–483 (1941); these Rev. **3**, 3] relating to Markov chains and obtained for compact spaces can be generalized to the noncompact case. The paper contains results on sets of points, defined by asymptotic properties of the flow, which have measure equal to the measure of the whole phase space in every possible invariant measure. *S. Ulam.*

Kolmogoroff, A. N. Stationary sequences in Hilbert's space. Bolletin Moskovskogo Gosudarstvenogo Universiteta. Matematika **2**, 40 pp. (1941). (Russian) [MF 9565]

Theorems on stationary sequences in Hilbert space are proved, which when applied to the theory of probability become theorems on linear interpolation and extrapolation in stationary sequences of chance variables. The theorems are discussed in their probability form in [Bull. Acad. Sci. URSS Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 3–14 (1941); these Rev. **3**, 4] where the theorems are stated.

J. L. Doob (Washington, D. C.).

Zasuhin, V. On the theory of multidimensional stationary random processes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 435–437 (1941). [MF 9630]

The author extends Kolmogoroff's theorems on linear interpolation and extrapolation in stationary stochastic processes (cf. the preceding review) to N-dimensional processes. If $N>1$, the determinant of the spectral functions plays the role of the spectral function in the case $N=1$.

J. L. Doob (Washington, D. C.).

Calculus of Variations

Mammana, Gabriele. Il minimo assoluto in taluni classici problemi di calcolo delle variazioni. III. Anais Acad. Brasil. Ci. **15**, 257–280 (1942). [MF 9361]

Curves of Ribaucour, which are integrals of the Euler equation $nyy''=1+y'^2$ for the functional

$$J_L=\int_{P_1}^{P_2}{}^{(L)} y^{1/n}(dx^2+dy^2)^{\frac{1}{2}}, \qquad n\neq 0,$$

L in the upper half-plane, were investigated regarding relative minima and the existence of an absolute minimum in the previous parts I and II [same Anais **14**, 53–77, 167–185 (1942); these Rev. **4**, 200]. For a fixed P_1 and a fixed n, the author now determines the locus of all points P_2 to the right of P_1 for which the absolute minimum of J_L is attained on a curve of Ribaucour, and the locus for which this minimum is given by the broken-line (two ordinates and segment of x-axis) solution of Goldschmidt. For $n<0$ and for $n\geqq 1$, the results follow the patterns of the familiar classical cases $n=-2$, -1 and $n=1$, 2, respectively. For $0<n<1$, the situation is somewhat different. For $n<0$, a curve of Ribaucour is the only extremal and accordingly gives an absolute minimum. For a fixed $n\geqq 1$ and for a fixed P_1 : (x_1, y_1), the curves of Ribaucour have an envelope Γ which extends from $(x_1, 0)$ to $(+\infty, +\infty)$. Let B be the domain in the upper half-plane bounded by $x=x_1$ and Γ, and let $x_2>x_1$. If P_2 : (x_2, y_2) is not in E, then the Goldschmidt solution is the only extremal and therefore gives an absolute minimum. On each curve Γ_θ of Ribaucour, through P_1 and with inclination θ, $-\pi/2<\theta<\pi/2$, there is a point P_2^* between P_1 and the point of tangency of Γ_θ and Γ, at which equal values of J_L occur on Γ_θ and on the broken line. The locus of points P_2^* is a curve I which extends from $(x_1, 0)$ to $(+\infty, +\infty)$. For P_2 in the part of E bounded by $x=x_1$ and I, the absolute minimum is attained on Γ_θ, while in the part of E bounded by I and Γ the absolute minimum is attained on the broken line. For $0<n<1$, Γ and I have vertical asymptotes and extend from $(x_1, 0)$ to $(x_1+r, +\infty)$ and $(x_1+s, +\infty)$, respectively, with $0<s<r<+\infty$. There is a first Γ_Θ, $-\pi/2<\Theta<\pi/2$, asymptotic to I; Γ_θ does not cut I for $\Theta\leqq\theta<\pi/2$. Thus if $x_2>x_1$ and $n\geqq 1$ are fixed, for sufficiently large y_2 the absolute minimum is attained on a curve of Ribaucour; but for $x_2>x_1+r$ and $0<n<1$, for all $y_2\geqq 0$ the only extremal is the broken line of Goldschmidt.

E. F. Beckenbach (Austin, Tex.).

Savage, L. J. On the crossing of extremals at focal points. Bull. Amer. Math. Soc. **49**, 467–469 (1943). [MF 8401]

The definition of focal points in the calculus of variations is analytical; it is in terms of a determinant becoming zero. The primordial focal point, however, was the center of curvature of a curve, where neighboring normals to the curve begin to ram one another. Morse and Littauer showed how to prove that the geometric idea of a focal point as the point where transitive geodesics neighboring to each other cease to form a field is still valid in problems of the calculus of variations which are analytic. In the present paper the author shows how this treatment may be modified to prove that the geometric idea is still valid in a Finsler space with three derivatives continuous. The opinions expressed here are those of the reviewer and not necessarily those of the Navy Department or the Naval Service at large.

C. B. Tompkins (Santa Barbara, Calif.).

Morse, Marston. Functional topology. Bull. Amer. Math. Soc. **49**, 144–149 (1943). [MF 7998]

The basic general requirements for getting results in the calculus of variations in the large are well established, but no new problem seems to fit the past explicit statements of these requirements, and for each problem a little shifting and fitting is necessary. In this paper the author prepares to extend the theory to nonregular problems. He proves the standard existence theorem for homotopic critical points in a space of curves joining two fixed points of an underlying space; in the curve space three separate metrics (Fréchet distance, Fréchet distance increased by the absolute value of the difference of lengths of the curves and Fréchet distance increased by the absolute value of the difference of an integral J along the curves) replace the single metric found in the original theory, and they are played against each other to get the standard existence theorem into a form which seems applicable to the problems for which it is intended. The paper is technical and not expository in nature. The opinions expressed here are those of the reviewer and not necessarily those of the Navy Department or the Naval Service at large.

C. B. Tompkins.

TOPOLOGY

*Bourbaki, N. **Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre III. Topologie générale. Chapitres III et IV.** Actual. Sci. Ind., no. 916. Hermann & Cie., Paris, 1942. iv+158 pp.

Chapters I and II of Livre III of the Bourbaki treatise [these Rev. **3**, 55] contained a development of the basic ideas of general topology centered around two concepts: (1) filters and their convergence (replacing sequences and sequential convergence); (2) uniform spaces (replacing metric spaces). The enormous degree of generality thus obtained is now utilized highly successfully in chapter III entitled "Groupes topologiques (théorie élémentaire)." Since the algebraic theory of groups will be taken up in Livre II, the chapter begins with the definition of a topological group. No separation axiom is assumed so that factorization by nonclosed subgroups is permissible. In discussing mappings of one group into another a careful distinction is drawn between the representations and homomorphisms, the first being merely continuous and multiplicative while the others in addition have to carry open sets into open sets of the image group. By attaching to each group a (left and a right) uniform structure a discussion of completeness in topological groups is made possible. The main theorem asserts that a topological separated (that is, Hausdorff) group can be completed if and only if

the mapping $x \to x^{-1}$ has a certain additional continuity property. It is not known whether this condition is always satisfied. It is satisfied in the commutative case. The next topic is the summability of infinite (uncountable) families in Abelian groups based on the Moore-Smith [Amer. J. Math. 44, 102–121 (1922)] definition of convergence. In a complete group every subfamily of a summable family is summable and associativity of the sum is secured. Summability is then neatly tied up with convergence and commutative convergence (that is, convergence under any rearrangement) of countable series. The topics of topological rings and fields are briefly discussed.

Chapter IV is entitled "Nombres réels." The ordered group Q of rational numbers is taken for granted, as it will be defined in Livre II. Using open intervals, Q is topologized. The group R of real numbers is defined as the completion of the group Q. It is then shown that the order structure and the multiplication can be continued from Q to R. The usual topological properties of R, like connectedness and the compactness of bounded closed subsets, are established. Numerical functions (that is, functions with real values defined over any set of arguments) and their continuity are studied next. Using the concept of a summable family in an Abelian group the authors quickly obtain the basic results concerning infinite series and products. The mechanism of approximation of real numbers by means of rationals is exhibited by means of the development of a real number according to a base. Using 2 as a base it is shown that the cardinal number of R is that of the family of infinite sequences of integers.

Each chapter concludes with a historical note. The one following chapter IV is particularly extensive and complete. The two chapters contain about 200 very instructive and skillfully chosen exercises. *S. Eilenberg.*

Bourgin, D. G. Linear topological spaces. Amer. J. Math. 65, 637–659 (1943). [MF 9391]

The paper is concerned with a real linear topological space L in the sense of von Neumann and Hyers and its adjoint space L^* of all linear functionals on L. Some of the results of this paper appeared previously [Proc. Nat. Acad. Sci. U. S. A. 27, 539–544 (1941); these Rev. 3, 137]. Among the results established in the second section is the theorem that, if a convex set K is properly contained in L and if it contains interior points, then it is regularly L convex in the sense that if $x_0 \bar{\in} K$ there is an $f_0 \in L^*$ such that $f_0(x_0) > \sup_K f_0(x)$. Many of the important results of sections 3 through 6 have been indicated in the review cited above. It is shown in section 6 that, if A is a bounded open set in L and if L is locally bounded, the quasi-norm satisfies $\|x_1+x_2\| \leqq b(\|x_1\|+\|x_2\|)$ for some $b \geqq 1$ depending on A and L. The number b is called the multiplier of the quasi-norm. It is also shown that, if L is locally bounded, then there is a continuous quasi-norm. Defining b_L as inf $(b_A | A$ bounded and open in $L)$, the author shows in the seventh section by an example that it is not always a possible multiplier. In the last section the $w^*(L^*)$ topology is considered. A set K in L^* is regularly L^* convex if for $f_0 \bar{\in} K$ there is an $x_0 \in L$ such that $f_0(x_0) > \sup_K f(x_0)$. The statement (a) K is regularly L^* convex is equivalent to (b) K is convex and $w^*(L^*)$ closed. If one assumes that L^* contains more than one point, a number of interesting results are obtained. If L is locally bounded f_n converges $w^*(L^*)$ to f_0 if and only if $\|f_n\| \leqq M$ and $f_n(x)$ converges pointwise on a dense set in L. Moreover, if S^* is the unit sphere $\|f\| \leqq 1$, then S^* is bicompact in the $w^*(L^*)$ topology. Banach's notion of transfinite limits is carried over when L is locally bounded and it is shown that if K in L^* is bounded then K is $w^*(L^*)$ closed and convex if and only if K is transfinitely closed.

H. H. Goldstine (Aberdeen, Md.).

Petroff, A. A. Sur les recouvrements des espaces compacts. Rec. Math. [Mat. Sbornik] N.S. 12(54), 109–120 (1943). (Russian. French summary) [MF 8797]

Translation of French summary is as follows. Let M be a compact set and let $M = \sum_{i=1}^{k} A_i$ be a covering of M by means of a finite number of closed sets. Consider the multiplicative covering of M by means of the sets $A_1, A_2, \cdots, A_k$ and all their nonempty products. Let $A_{i_1}A_{i_2}\cdots A_{i_r}$ be an element (that is, a nonempty product of A's) and take a strictly decreasing sequence of elements $A_{j_1} \supset A_{j_1}A_{j_2} \supset \cdots \supset A_{j_1}A_{j_2}\cdots A_{j_n} = A_{i_1}A_{i_2}\cdots A_{i_r}$. The largest number of terms in such a sequence is called the true order of the element $A_{i_1}A_{i_2}\cdots A_{i_r}$. The order of $A_{i_1}\cdots A_{i_r}$ is defined as the number of A's containing $A_{i_1}\cdots A_{i_r}$. This paper investigates the connection between order and true order of an element. *M. Kac* (Ithaca, N. Y.).

Fan, Ky. Sur les ensembles possédant la propriété des quatre points. C. R. Acad. Sci. Paris 213, 518–520 (1941). [MF 9172]

If a, b, c and d are four distinct points of a connected space E, then the pair a, b is said to separate the pair c, d (in symbols $ab \| cd$) if $E-(a)-(b)$ contains no component containing c and d. A space E is said to have the property of four points if it is connected, contains more than one point and if for any four distinct points a, b, c, d one of the following is true: $ab \| cd$, $ac \| bd$, $bc \| ad$. The author announces two results, one of which is that a necessary and sufficient condition for a space E of Riesz to be homeomorphic to an arc, a ray, a line or a circumference is that E be separable, locally connected and possess the property of four points. *D. Montgomery* (Princeton, N. J.).

Arnold, B. H. Decompositions of a T_1 space. Bull. Amer. Math. Soc. 49, 768–778 (1943). [MF 9321]

The author considers a topological space S in which single points are closed and defines a decomposition of S as a partition of S into disjoint, closed and nonempty subsets. Such a decomposition is upper semi-continuous in case to each set X in the decomposition and each open set $U \supset X$ there corresponds an open set $V \supset X$ such that any set of the decomposition meeting V is in U. The set M of upper semi-continuous decompositions D is partially ordered by saying that if each set of D' is in some one of D'', then $D' < D''$. It is then shown that two T_1 spaces are homeomorphic if and only if their sets of upper semi-continuous decompositions are isomorphic. If S is a bicompact Hausdorff space, then M is a complete lattice; it is also proved that, when S is metric, M is a complete lattice if and only if S is either discrete or bicompact. The author characterizes separation and connectedness properties of the space in terms of order properties of M and closes the paper with some interesting examples bearing on his theorems.

H. H. Goldstine (Aberdeen, Md.).

Franz, Wolfgang. Abbildungsklassen und Fixpunktklassen dreidimensionaler Linsenräume. J. Reine Angew. Math. 185, 65–77 (1943). [MF 9589]

It is proved that two mappings f and g of a lens space (m, q) into a lens space (m, κ) are homotopic if and only if

they have the same degree and induce the same homomorphism of the fundamental groups. The proof uses the construction of the induced "covering" mappings f^* and g^* of the universal covering spaces (=3-spheres) of (m, q) and (m, κ). To effect this construction the author proves a general theorem about the existence of induced "covering" mappings between covering spaces. All possible homotopy classes are determined. For a mapping of (m, q) into itself, the fixed point index and the number and indices of the fixed point classes are given in terms of the degree and the endomorphism of the fundamental group. In the proof of this result Reidemeister's "Homotopiekettenring" is used.

H. Samelson (Syracuse, N. Y.).

Fox, Ralph H. On fibre spaces. II. Bull. Amer. Math. Soc. **49**, 733–735 (1943). [MF 9314]

A converse of the covering homotopy theorem for fiber spaces is proved; it is shown that, if for a mapping π of a space X into a space B the covering homotopy theorem holds (every homotopy in B, the initial value of which is of the form πf (is a "projection"), can be covered by a homotopy in X the initial value of which is f), then π is a fiber mapping of X into B (in the sense of Hurewicz-Steenrod). The space B is assumed to be "locally equiconnected," which means, roughly speaking, that sufficiently nearby points are connected by paths which depend continuously on the endpoints; compact ANR-sets have this property. A new proof is then given of Borsuk's extension theorem in the form of a fiber theorem in function spaces: let A be a closed subset of the space X, and let Y be a compact ANR-set; then the "sectioning" operation π, which sends each mapping $f \in Y^X$ into the partial mapping $f|A$, is a fiber mapping of Y^X into Y^A. *H. Samelson* (Syracuse, N. Y.).

Leray, Jean. Les complexes d'un espace topologique. C. R. Acad. Sci. Paris **214**, 781–783 (1942). [MF 9450]

Leray, Jean. L'homologie d'un espace topologique. C. R. Acad. Sci. Paris **214**, 839–841 (1942). [MF 9457]

Leray, Jean. Les équations dans les espaces topologiques. C. R. Acad. Sci. Paris **214**, 897–899 (1942). [MF 9462]

Leray, Jean. Transformations et homéomorphies dans les espaces topologiques. C. R. Acad. Sci. Paris **214**, 938–940 (1942). [MF 9466]

The notes constitute an outline of a course in topology. In the first two notes the cohomology theory and the homology theory in topological spaces are outlined. The outline is very concise with no proofs and not very complete definitions but it seems likely that the cohomology theory developed will coincide with that of Čech (at least in the case of bicompact spaces). In the remaining two notes the author uses his homology theory to outline the topological principles that lead to existence theorems for solutions of certain nonlinear equations in Banach spaces [see J. Leray and J. Schauder, Ann. Sci. École Norm. Super. **51**, 45–78 (1934)]. *S. Eilenberg* (Ann Arbor, Mich.).

Steenrod, N. E. Homology with local coefficients. Ann. of Math. (2) **44**, 610–627 (1943). [MF 9400]

The author introduces the concept of a system of local groups in a (arcwise connected) space R. It consists of two functions: one associates with each point $x \in R$ a group G_x, the other associates with each homotopy class α_{xy} of paths joining two points x and y an isomorphism $\bar{\alpha}_{xy} : G_y \to G_x$ in such a way that $\overline{\alpha_{xy}\beta_{yz}} = \bar{\alpha}_{xy}\bar{\beta}_{yz}$. [In the paper the inverse isomorphisms $G_x \to G_y$ were used; however the author has realized that this led to confusions in the notation.] If the attention is centered at one point $x \in R$ it follows that the fundamental group $F_x(R)$ of R with x as base point acts as a group of left operators on G_x. It is shown that this determines the whole local system to within an isomorphism. If F_x operates trivially on G_x, the local system is called simple, and is isomorphic with a system in which all the groups G_x are identical and all the $\bar{\alpha}_{xy}$ are identities. Typical examples of local systems of groups are the system of fundamental groups $F_x(R)$ and also the homotopy groups $\pi_n(R, x)$ with various points $x \in R$ as base points.

A local system of Abelian groups $\{G_x\}$ is used as "coefficients" for the construction of homology and cohomology groups. The theory is first developed for a complex and then carried over to a space using either singular or Čech cycles. If R is a complex, then a chain with coefficients in $\{G_x\}$ is a function which to each cell σ (of a suitable dimension) associates an element of $G_{x(\sigma)}$, where $x(\sigma)$ is a point of σ picked in advance. Using the isomorphisms $\bar{\alpha}_{xy}$ the boundary and coboundary of a chain over $\{G_x\}$ can be defined and are again chains over $\{G_x\}$. Roughly speaking, a cycle (or cocycle) with local coefficients is locally a cycle (or cocycle) in the ordinary sense. Consequently any operation on ordinary chains which is the logical sum of local operations can be carried over to chains with local coefficients. Following this idea the theory of multiplications in a complex is carried over. Passing to manifolds, it is shown how a suitable use of local coefficients permits the establishment of an intersection theory and leads to a Poincaré duality theorem in nonorientable manifolds. It is further shown that (after a suitable interpretation) the theory includes Reidemeister's concept of "Überdeckung," this way adding clarity to Reidemeister's work and implying the invariance of Reidemeister's group of "Homotopieketten." The theory was already applied in an earlier paper of the author [Ann. of Math. (2) **43**, 116–131 (1942); these Rev. **3**, 144]. *S. Eilenberg* (Ann Arbor, Mich.).

Bockstein, M. A complete system of fields of coefficients for the ∇-homological dimension. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 187–189 (1943). [MF 9241]

Having translated the concept of the Alexandroff homology dimension into the language of cohomology and using the results and methods of a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 243–245 (1942); these Rev. **5**, 48], the author states that, if the cohomology dimension is known for the following four classes of groups: (a) the group of rationals, (b) the group of rationals whose denominator is not divisible by the prime p, (c) the cyclic groups of prime order p, (d) the group of rationals whose denominators are powers of a prime p, then the cohomology dimension is known for any coefficient group.

S. Eilenberg (Ann Arbor, Mich.).

Eckmann, Beno. Stetige Lösungen linearer Gleichungssysteme. Comment. Math. Helv. **15**, 318–339 (1943).

The author considers a system $\sum_{k=1}^{n} a_{ik}x_k = 0$ of r homogeneous equations with n unknowns ($n > r$), where the coefficients a_{ik} are continuous real valued functions defined on a space R and the matrix $\|a_{ik}\|$ has rank r for every point of the space R. The problem is to find as many as possible independent solutions of the system, the solutions also being continuous functions of R. After orthogonalization the problem takes the following form: given an orthogonal

matrix $\|a_{ik}\|$ with n columns and $r<n$ rows whose terms a_{ik} are continuous real valued functions on a space R, what is the maximal number l^* of rows by which the matrix can be increased preserving orthogonality and continuity. The author then considers the manifold $V_{n,r}$ of all orthogonal numerical matrices with n rows and r columns. The system of matrices studied above becomes then a continuous function $f: R \to V_{n,r}$. A fiber mapping $P: V_{n,r+j} \to V_{n,r}$ is obtained by erasing the last j rows in each matrix of $V_{n,r+j}$. The problem of enlarging the matrices by adding j rows becomes the problem of finding a mapping $\bar{f}: R \to V_{n,r+j}$ such that $P\bar{f}=f$. In this purely topological form the problem is solved in a great many special cases, using, in some cases, powerful methods of the homotopy group theory.

S. Eilenberg (Ann Arbor, Mich.).

Rey Pastor, J. Jordan's theorem for closed polyhedral varieties. Revista Union Mat. Argentina **9**, 89–95 (1943). (Spanish) [MF 9440]

The author proposes a proof that a closed Jordan polygon separates the plane, and a generalization to polyhedra in n-space. The proof in the plane depends on drawing parallels to a fixed direction through the vertices of the polygon; between each two parallels the sides of the polygon bound a sequence of trapezoids and triangles, to which alternate + and − signs can be ascribed. The trapezoids and triangles of the same sign fit together to form the two regions in question. The reviewer fails to see how the author uses the fact that the polygon is simple, and avoids appealing to intuition in showing that there are not more than two regions. *R. P. Boas, Jr.* (Cambridge, Mass.).

GEOMETRY

Kerawala, S. M. Euclidean aspect of Hjelmslev's geometry. Proc. Benares Math. Soc. (N.S.) **3**, 39–53 (1941). [MF 9437]

Let the accuracy of locating a point in the plane be determined by a constant δ, so that a physical point determined by geometric conditions lies in a domain of diameter not greater than δ with area not greater than $\pi\delta^2/4$ with an abstract point P as center of gravity. Such a domain is called the pointoid (P). Linoid, angloid, etc., are defined similarly. If exact methods are applied to these concepts the following results are found. The intersection of two linoids which intersect at angloids of average values between 60° and 90° contains at least one common pointoid, but if the angloid is below 10° then the intersection contains at least 7 disjoint pointoids (independently of the accuracy δ!). A circloid of average radius a has with a tangentoid at least $16\cdot 2^{\frac{1}{2}}3^{-\frac{1}{2}}a^{\frac{1}{2}}\delta^{-\frac{1}{2}}$ disjoint common pointoids, a number which tends to ∞ as $\delta \to 0$. This makes it doubtful whether "refinement of the instruments" leads to the axioms of exact geometry. *H. Busemann* (Chicago, Ill.).

Buck, R. C. Partition of space. Amer. Math. Monthly **50**, 541–544 (1943). [MF 9529]

It is a known fact that n straight lines in general position divide the Euclidean plane E_2 into $1+n+n(n-1)/2$ regions and divide the projective plane into $1+n(n-1)/2$ regions. This fact is generalized by the author, who gives formulas for the number $M_r(p, n)$ of p-cells determined by the intersections of hyperplanes in general position in Euclidean r-space E_r. These expressions are sums of terms which are products of two binomial coefficients. Use is made of the number of bounded regions of E_r in making a similar count for projective r-space, since the unbounded regions are joined in pairs in projective space. *J. S. Frame.*

Behrend, F. A. A polyhedral model of the projective plane. J. Proc. Roy. Soc. New South Wales **77**, 20–23 (1943). [MF 9536]

The author improves upon the work of Merz and Humbert [Comment. Math. Helv. **14**, 134–140 (1942); these Rev. **3**, 299] by constructing a simpler model of the projective plane in the form of a polyhedron in ordinary space without any singular vertices. There are 6 vertices of degree 3, 6 of degree 4, 21 edges, 9 quadrangular faces and one hexagonal face. Six of the quadrangular faces penetrate one another along "false edges." Adequate instructions are given for carrying out the construction in paper or thin cardboard.

H. S. M. Coxeter (Toronto, Ont.).

Abramescu, N. Sur la relation entre les origines de la géométrie Cayleyenne et la géométrie sphérique. Bull. Sci. École Polytéch. Timişoara **11**, 3 pp. (1943). [MF 9422]

It is shown that the generators of a sphere S at a point P are the tangents from P to the isotropic cone with vertex at O the center of S. By projecting S from O onto a plane π one obtains points and lines so that the arc length of great circles and the angles between curves on S can be expressed in the fashion of Cayley in terms of the logarithms of certain cross ratios involving these points and lines.

V. G. Grove (East Lansing, Mich.).

Abramescu, Nicolas. Sur les courbes anallagmatiques. Acad. Roum. Bull. Sect. Sci. **25**, 113–115 (1942). [MF 9514]

A characteristic property of a curve which transforms into itself by inversion with respect to a circle is the following: a curve (E) which is invariant under inversion is the envelope of a circle (C) whose center describes a given curve and which cuts a given fixed circle (the circle of inversion) orthogonally. Let the points of contact of (C) with (E) be A and B. This paper gives a synthetic and an analytic proof of the theorem: the line joining the centers of curvature of (E) at A and B passes through the center of inversion. [For further literature see Laguerre, Oeuvres, vol. 2, Gauthier-Villars, Paris, 1905, pp. 41–45, 54–58, 64–72; Gomes Teixeira, Traité des courbes spéciales remarquables planes et gauches, vol. 1, Imprimérie de l'Université, Coimbre, 1908, pp. 68, 236, 241; H. Brocard and T. Lemoyne, Courbes géométriques remarquables . . . , vol. 1, Vuibert, Paris, 1919, p. 2.] *V. G. Grove.*

Schönberg, Mario. On an extension of the spinor calculus. II. Anais Acad. Brasil. Ci. **15**, 97–108 (1943). (Portuguese) [MF 9359]

The author generalizes the work of his first communication [same Anais **13**, 129–144 (1941); these Rev. **3**, 252] to spaces of higher dimensionality, following closely the methods of E. Cartan. *A. Schwartz.*

Convex Domains, Integral Geometry

v. Sz. Nagy, Gyula. Ein Beweis des Vierscheitelsatzes. Jber. Deutsch. Math. Verein. **52**, 198–200 (1942). [MF 9052]

A lune is the sum of two convex arcs that have their endpoints A, B but no other points in common and that lie on the same side of the straight line AB. If the two arcs are continuously curved and if their total curvatures are not greater than π, the lune is called simple. It then lies entirely between two suitable parallel straight lines through A and B. By moving either of the arcs parallel to these lines until it touches the other for the last time, we obtain a lemma of Brunn [Über Ovale und Eiflächen, Inaug. Diss., Munich, 1887, p. 7]: the maximum curvature of the exterior arc of a simple lune is not smaller than the minimum curvature of its interior arc.

Let the curve C be closed, convex and continuously curved. Let $\kappa(P)$ denote the curvature of C at P. Let the circle L of radius $1/\kappa_0$ intersect C in $2n$ points $P_1, \cdots, P_{2n}=P_0$ which divide the figure into simple lunes. Let the arcs $P_1, P_2, P_3, P_4, \cdots$ ($P_0P_1, P_2P_3, \cdots$) of C lie outside (inside) of K. From Brunn's lemma, there are points $Q_\lambda \subset P_\lambda P_{\lambda+1}$ such that $\kappa(Q_{2\lambda+1}) \leqq \kappa_0$ and $\kappa(Q_{2\lambda}) \leqq \kappa_0$. Hence $\kappa(P)$ has at least one minimum (maximum) between $Q_{2\lambda-1}$ and $Q_{2\lambda+1}$ ($Q_{2\lambda}$ and $Q_{2\lambda+2}$). Altogether $\kappa(P)$ has at least $2n$ extremums. If we choose for K the circumcircle of an acute triangle inscribed in C, our assumptions are satisfied and $2n \geqq 4$ yields the four vertex theorem. *P. Scherk.*

Combes, Bernard. Sur le plus petit corps convexe contenant un arc de courbe dans l'espace à n dimensions. C. R. Acad. Sci. Paris **215**, 291–293 (1942). [MF 9499]

This note deals with the convex hull of an arc of a curve in $x_1, \cdots, x_n$-space, intersected by any hyperplane in at most n points. Special results on the behaviour of that hull for $n \to \infty$ are obtained for the curve given by

$$x_p = x^p \text{ for } p=1, \cdots, n \quad (a \leqq x \leqq b).$$

Applications to moment problems are indicated. No detailed proofs are given. *F. John* (Aberdeen, Md.).

Bol, G. Zur Theorie der Eikörper. Jber. Deutsch. Math. Verein. **52**, 250–266 (1942). [MF 9057]

Let B be a 3-dimensional convex body. For any point P of B let the minimal distance of P from the boundary of B be denoted by a_P. This paper is concerned with inequalities for the "moments" $I_\mu = I_\mu(B) = \int_B a^\mu dv$. The author proves that, for $k \geqq 1$,

$$(1) \qquad I_k^3 - (1-1/(1+k)^2) \cdot I_{k-1} \cdot I_{k+1} \leqq 0,$$

$$(2) \qquad I_k^3 - (1-1/((k+1)(k+3))) \cdot I_{k-1} \cdot I_{k+1} \geqq 0,$$

and that

$$(3) \qquad V^2 - 2OI_1 \leqq 0,$$

$$(4) \qquad V^2 - (4/3)OI_1 \geqq 0,$$

where V is the volume, O the surface area of B.

The proofs of these inequalities make use of some previous results of the author [Abh. Math. Sem. Hansischen Univ. (1942)], which concern a generalization of the notion of "parallel surfaces." The set of points of B with $a_P \geqq h$ is called the "interior parallel body" to B of distance h; B_h is a convex body defined for positive h not exceeding the radius of the largest inscribed sphere, and can be defined for negative h as the convex body bounded by the ordinary (exterior) parallel surface of distance $-h$. Inequalities (2), (4) are equivalent to the statement that the expressions $(I_k(B_h))^{1/(k+3)}$ are concave functions of h; they are proved by induction over k, starting from an inequality of the Brunn-Minkowski type. Inequalities (1) and (3) are simple consequences of the logarithmic convexity of I_μ as a function of μ. Analogous inequalities are derived for the expressions $I_{G,\mu}(B)$, formed by defining in a suitable manner a function a_G of the straight lines G meeting the body B. *F. John.*

Schmidt, Erhard. Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raum jeder Dimensionenzahl. Math. Z. **49**, 1–109 (1943). [MF 9191]

This paper is a continuation of the following papers: (a) Math. Z. **44**, 689–788 (1939); (b) **46**, 204–230 (1940); (c) **46**, 743–794 (1940) [cf. these Rev. **2**, 12, 262]. In it the author completes the extension to hyperbolic and elliptic spaces (of constant curvature) of his proof, given in (a) for Euclidean space of n dimensions, of the isoperimetric property of the sphere, namely, that it has a smaller "surface area" than any other "regular" body of equal volume. (A body is "regular" if, roughly speaking, "die Berandung aller Dimensionenzahlen aus endlich vielen zweimal stetig differenzierbaren Stücken besteht" [details in (a), §§5 and 6].) Papers (a), (b) and (c), of which the last two dealt with hyperbolic and elliptic space, respectively, contained proofs of two special cases of the isoperimetric property: (i) the case $n=2$, (ii) the case where (in n dimensions) only bodies of revolution are admitted. The author now reduces the general case to cases (i) and (ii) by an induction. For this purpose he adapts the "rounding" process used by Schwarz for Euclidean 3-dimensional space: the given body (in n dimensions) is replaced by a body of revolution, about a suitably chosen axis, the "area" of each cross-section being unchanged. In the Euclidean case the sections were made by hyperplanes perpendicular to the axis; but for each non-Euclidean case the author finds it necessary to use instead a family of equidistant "surfaces," orthogonal to and symmetrical about the axis and of constant Gaussian curvature. The volume is then still the integral of the cross-sectional "area," and the isoperimetric inequality, assumed for $(n-1)$-dimensional spaces of constant curvature, can still be applied to each section.

The author states that his results enable him to generalize certain refinements upon the isoperimetric inequality and also the Brunn-Minkowski theorem; he reserves the proof for subsequent publication. *H. P. Mulholland.*

Kasner, Edward and Supnick, Fred. The Apollonian packing of circles. Proc. Nat. Acad. Sci. U. S. A. **29**, 378–384 (1943). [MF 9767]

An analytic proof of the special case of Vitali's covering theorem proved geometrically in a note of E. Kasner [Scripta Math. **9**, 26 (1943); these Rev. **4**, 252]. *P. Franklin* (Cambridge, Mass.).

Fejes, L. Über die dichteste Kugellagerung. Math. Z. **48**, 676–684 (1943). [MF 9007]

The determination of the densest distribution of non-overlapping unit spheres in space is an unsolved problem, except under the assumption that the centers of the spheres form a space lattice. Under this assumption it is known that the lattice must be composed of rhombic dododecahedra, and the density is $D_0 = \sqrt{2}\pi/6 \approx 0.7405$. In general the density associated with a single sphere, center at P_i, is the ratio

of its volume to the volume of the interior of the polyhedron T_i, composed of those points which are nearer to P_i than to any other center. The density of the distribution is then taken as the limit of the average of the densities associated with those unit spheres which lie within a large cube W when the volume of W becomes infinite. It is shown in this paper that this distribution density cannot exceed 0.7546, which is the density associated with a single sphere when the polyhedron T_i is the circumscribed regular dodecahedron. But since space cannot be filled completely by regular dodecahedra, the average distribution density for the most dense space distribution must be less than this value. *J. S. Frame* (East Lansing, Mich.).

Algebraic Geometry

Apéry, Roger. Sur les quintiques à cinq rebroussements. C. R. Acad. Sci. Paris **213**, 674–676 (1941). [MF 9649]

The Del Pezzo quintic is a plane curve having five cusps. It is uniquely fixed when the positions of the five cusps are given. From the Plücker numbers the dual is of the same form. The present note shows that the curve is invariant under polarity as to a conic. It is of genus one. The point coordinates are expressed in terms of linear functions of the Weierstrass elliptic function of a parameter u and its first three derivatives. The dual coordinates are then expressed in the same form, from which the result follows.

V. Snyder (Winter Park, Fla.).

Terracini, Alejandro. The S_3 which are osculatory to curves of a variety and a new characterization of a class of varieties. Univ. Nac. Tucumán. Revista A. **3**, 317–339 (1942). (Spanish) [MF 9274]

The Veronese surface of S_5 has the two properties of being the only nonconical surface of S_r $(r \geqq 5)$ (1) the chords of which generate a variety of only four dimensions and (2) whose tangent planes meet two by two. In an earlier paper the author [Rend. Circ. Mat. Palermo **31**, 392–396 (1911)] generalized the premises by replacing V_4 by a V_{2k} in which the tangent S_k still meet two by two. For $k=2$, the Veronese surface differs from a cone since the V of the latter is of dimension 3. If $r=2k+1$, normal dimensionality exists; if $r<2k+1$, the system is called irregular. In the present paper properties (1) and (2) are replaced by (3) that is, the V_k of osculating S_k is irregular. Conversely, if (3) is satisfied, the osculating S_k of the system meet two by two.

Let A be a point of V_k of S_r and a a line through it; $S(m, 1)$ is the osculating S_3 having m point contact with V_k at A. This concept is a generalization of systems of asymptotic lines of a surface in S_3. The fundamental projectivity is defined and applied to generate V_k. In the normal case it is applied to the hyperosculating systems of curves of V through A. This is finally applied to an alternate approach to the system of differential equations of the irregular case and applied to the case of the Veronese surface. It can also be accomplished by the map of the Veronese surface. An earlier paper outlines this procedure [Univ. Nac. Tucumán. Revista A. **2**, 245–329 (1941); these Rev. **4**, 54].

V. Snyder (Winter Park, Fla.).

Haenzel, G. und Reutter, F. Die Geometrie der linearen Strahlenkongruenz. IV. Über algebraische Regelflächen vom Grade $2n$ mit zwei n-fachen Leitgeraden ($n=3, 4, 5, 6$). J. Reine Angew. Math. **185**, 78–101 (1943). [MF 9590]

This paper follows a series of earlier ones by one or the other of the present authors on closely related subjects [J. Reine Angew. Math. **166**, 167–181 (1932); **173**, 91–113 (1935); **175**, 169–181 (1936); **178**, 229–252 (1938)]. It begins with the consideration of a cubic polynomial in the complex variable z, its Hessian H and its Jacobian J; H is supposed to have two distinct roots, which are taken at the points ± 1. A circle in the z-plane is drawn, having these points on a diameter. Circles are drawn having the three roots of J for centers and orthogonal to the circle just mentioned. Consider the three transformations

$$w=f_3(z)/H(z), \quad w=J(z)/zH, \quad w=f_3(z)/J(z).$$

The first is applied to the pencil of lines about the origin; the result is a pencil of tricircular sextics. The orthogonal pencil of concentric circles is then transformed into a second pencil of tricircular sextics. The other transformations are then applied to the same spider web, and all subjected to inversion as to the circle about the roots of H as diameter. The same procedure is then applied to a quartic and to its covariants, resulting in orthogonal pencils of circular curves of orders 6 to 12. It is repeated for various forms of the defining quartic. Then the whole procedure is applied to the dual complex plane, defined by $z=x+\epsilon y$, in which $\epsilon^2=1$, with analogous results. All this is preparatory to the study of ruled surfaces contained in a linear line congruence with two separate directrices of multiplicity half that of the order of the surfaces. The curves already discussed are the sections of these surfaces by the infinite plane. The fundamental quadratic identity among the line coordinates is reduced by means of the congruence to a quadric surface in [3]. Surfaces of orders 6 to 12 are considered, but only as isolated examples, with no attempt at a classification. The cases discussed are to be regarded as samples of the kind of functions under consideration. *V. Snyder.*

Jones, R. R. Some manifolds generated by normal rational curves. Proc. Cambridge Philos. Soc. **39**, 153–159 (1943). [MF 9197]

Prime sections of the Bordiga surface F^6 in [4] can be represented by plane quartic curves on ten base points. The author considers the special F_L^6 for which the base points are the intersections of five lines. By projecting F_L^6 from one of its points into [3] a quintic surface is obtained. The author obtains properties of this quintic from known properties of F_L^6. The theorems are then generalized by considering the $(n+1)$ics on the $(n+1)(n+2)/2$ intersections of $n+2$ lines, and the F^r, $r=(n-1)(n+2)/2$, in $[n]$ into which they map the plane. *R. J. Walker.*

Differential Geometry

Roger, Frédéric. Sur la courbure des ensembles plans. C. R. Acad. Sci. Paris **214**, 852–854 (1942). [MF 9459]

Let E be a plane set and $p \in E$. Let $p' \neq p$ and $p'' \neq p$, p' be sequences of points in E such that (α) the rays $\overrightarrow{pp'}$ and $\overrightarrow{pp''}$ converge to the same ray δ (a semitangent of E at p); (β) a circular arc with origin p through p' and p'' exists whose radius converges to a number R and which subtends

a central angle that tends to π. After a proper agreement as to sign, $\pm 1/R$ is called a curvature of E in the direction of δ. Put $W(\delta)=\sup K(\delta)$ and $w(\delta)=\inf K(\delta)$, where $K(\delta)$ traverses all curvatures in the direction of δ. At every point $p \in \bar{E}-N$, where $N \subset \bar{E}$ has length 0 (that is, N is contained in a countable set of rectifiable curves c_i and $N \cap c_i$ has measure 0), the curvatures are distributed as follows: (1) $W(\delta)=\infty$ and $w(\delta)=-\infty$ in the direction of every semitangent δ of E at p; (2) there is a unique tangent $T=\delta+\delta'$ and either (a) $W(\delta)=\infty$, $w(\delta)=W(\delta')=$finite number, $w(\delta')=-\infty$ or (b) $W(\delta)=w(\delta)=W(\delta')=w(\delta')=$finite number. *H. Busemann* (Chicago, Ill.).

Mirguet, Jean. Sur une classe de surfaces à double courbure continue. C. R. Acad. Sci. Paris **213**, 201–203 (1941). [MF 9160]

"Les orthosurfaces T, dont le paratingent au moins second est partout réduit à un nombre fini de droites ont été signalées par M. Bouligand dès 1930; j'ai démontré en 1936 que leur contingent en chaque point est plan, dièdre ou polyèdre convexe. Le présent exposé établit l'existence et les conséquences d'une double courbure continue en certains points de ces surfaces." *Author's summary.*

Bouligand, Georges. Extension du théorème de Dupin. C. R. Acad. Sci. Paris **213**, 156–157 (1941). [MF 9158]

The theorem of Dupin that surfaces of a triply orthogonal system intersect one another in the lines of curvature of each is extended to surfaces for which certain derivatives of the position vector are not uniquely defined. *V. G. Grove.*

Hsiung, Chuan-Chih. Theory of intersection of two plane curves. Bull. Amer. Math. Soc. **49**, 786–792 (1943). [MF 9324]

Let C and $\bar{C}$ be two curves which intersect at an ordinary point O on each, the tangents to C and $\bar{C}$ at O being distinct. Canonical forms of the power series expansions of C and $\bar{C}$ are obtained by choosing for the triangle of reference the tangents to the given curves at O and the line joining the poles of these tangents with respect to conics having third order contact with C and $\bar{C}$ at O. Two projective invariants I and J are characterized in terms of cross ratios and the choice of unit point is made to depend on I and J. [Cf. also C. C. Hsiung, Duke Math. J. **10**, 539–546 (1943); these Rev. **5**, 76.] *V. G. Grove* (East Lansing, Mich.).

Abramescu, N. Sur la relation entre les quadriques de Moutard et les sections d'une surface par de plans menés par une droite passant par un point de la surface. Mathematica, Timişoara **19**, 16–18 (1942). [MF 9421]

It is shown that the locus Σ of the osculating conics of the curves of section of a surface S by planes through a line D through a point on S is ordinarily a surface of the ninth order. If D is a nonasymptotic tangent of S, Σ reduces to the quadric of Moutard associated with D.

V. G. Grove (East Lansing, Mich.).

Decuyper, Marcel. Directrices de Wilczynski. C. R. Acad. Sci. Paris **214**, 315–317 (1942). [MF 9503]

Consider a surface referred to its asymptotic net. Let M be an arbitrary point of the surface; M_1, M_2 the points where the second directrix of Wilczynski cuts the asymptotic tangents through M; M_3 the point where the first directrix cuts the quadric of Lie. First the author evaluates the invariants β, γ introduced by Fubini and Čech. Next the direction parameters of the first directrix are found in terms of β, γ. These direction parameters are shown to depend upon the fourth order partial derivatives of the surface equations. Finally, since the second directrix may be defined as the intersection of the tangent plane at M with the plane OM_1M_2, the equation of this latter plane is determined. *N. Coburn* (Austin, Tex.).

Badaljan, V. H. Caractéristique géométrique des congruences à invariant de Waelsch constant. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 339–341 (1941). [MF 9566]

This paper is devoted to proving the following theorem: a necessary and sufficient condition that a rectilinear congruence (with focal surfaces (M_1), (M_2)) have constant Waelsch invariant is that there exist two straight lines M_1M_4, M_2M_3 associated with each element M_1M_2 which are reciprocal polars with respect to the Darboux quadrics adjoint to the homologous points of $(M_1)(M_2)$. The points M_1 to M_4 form vertices of a canonical tetrahedron at each point of (M_1) and the components of infinitesimal projective displacement between them are exhibited for use in the ensuing demonstration. Then the assumption that M_1M_4, M_2M_3 are reciprocal polars of a second order osculating quadric Q of (M_1) at M_1 and that (M_1) and Q intersect in the three Darboux directions is found to imply constancy of Waelsch invariant. A retracing of steps proves necessity.

J. L. Vanderslice (College Park, Md.).

Pa, Chenkuo. A transformation of Jonas surfaces. Bull. Amer. Math. Soc. **49**, 793–796 (1943). [MF 9325]

This paper is concerned with the projective differential geometry of Jonas surfaces in Euclidean three space. In particular, the author proves the theorem: "the projection in a fixed plane of a Jonas net of a Jonas surface is a plane net with equal point invariants and represents the projection of the asymptotic net of another Jonas surface." As a special case of this theorem, the author obtains the result: "any Jonas net of a Jonas surface S is in a fixed point perspective to the asymptotic net of another Jonas surface Q; conversely, if a Jonas net of a Jonas surface is in perspective to the asymptotic net of another surface Q, then Q is also a Jonas surface." *N. Coburn* (Austin, Tex.).

Pa, Chenkuo. On the surfaces whose asymptotic curves of one system are projectively equivalent. Univ. Nac. Tucumán. Revista A. **3**, 341–349 (1942). [MF 9275]

A. Terracini [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **59**, 451–461 (1924)], W. Blaschke [Abh. Math. Sem. Hansischen Univ. **4**, 225–231 (1926)] and E. P. Lane [Bull. Amer. Math. Soc. **46**, 117–120 (1940); these Rev. **1**, 170] have shown that, if the asymptotic curves of one system on a nonruled surface belong to linear complexes, then the asymptotic curves of the system are projectively equivalent. First the author offers a new proof of this theorem. Secondly he considers the converse theorem. The validity of the converse for a ruled surface has been shown by G. Fubini [Bull. Amer. Math. Soc. **47**, 448–451 (1941); these Rev. **3**, 16]. It is shown by the author that the converse for a nonruled surface furnishes (1) surfaces whose asymptotic curves of one or both systems belong to linear complexes, (2) ∞^2 projective minimal surfaces of coincidence. *N. Coburn.*

Pa, Chenkuo. The projective theory of surfaces in ruled space. I. Amer. J. Math. **65**, 712–736 (1943). [MF 9397]

In the first portion of the paper the author derives various invariant quadric surfaces and curves projectively

connected with a point P of a surface S in ordinary three dimensional Euclidean space. To determine these invariants, the author introduces the asymptotic ruled surface Q_u^h (and Q_v^h) composed of the tangents to the asymptotic curves $v=$constant (and $u=$constant) at the points of a curve C_u^h (and C_v^h) of S. Here u and v refer to the surface parameters of the asymptotic net on S; C_u^h passes through P and touches the asymptotic tangent at P. The fundamental quantities associated with Q_u^h, the equation of the Moutard quadric associated with Q_u^h and various Darboux quadrics are found. Furthermore, generalizations of Darboux's pencil of quadrics and the Bompiani-Lane principal quadrics are obtained. In the second portion of the paper the equation of the osculating linear complex of every asymptotic osculating ruled surface is found. A theorem of Sullivan [Trans. Amer. Math. Soc. 15, 167–196 (1914)] is improved and some new invariant quadrics analogous to the Lie quadric are defined. The paper concludes with a brief study of linear complexes. [It should be understood that the above review covers merely the highlights of a paper which contains a superabundance of detailed results.]

N. Coburn (Austin, Tex.).

Rischkov, V. Sur la déformation à réseau conjugué persistant contenant une famille de lignes coniques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 288–289 (1941). [MF 9572]

The results of Egoroff's detailed investigation of conjugate nets which include one or two families of cone curves [Rec. Math. [Mat. Sbornik] **28**, 167–187 (1911)] only apply if the vertices of cones circumscribed along a family of cone curves lie on a plane curve. This condition is known to hold when both families of the net are cone curves. The author shows that the same is true when only one family are cone curves, thus establishing the validity of Egoroff's results in the general case. The differential equations of the surface of the net are written in plane coordinates, together with the conjugacy condition and a condition that one of the coordinate curve families be cone curves. On the hypothesis that the curve of vertices is nonplanar these equations are shown to have no solution. *J. L. Vanderslice.*

Strubecker, Karl. Differentialgeometrie des isotropen Raumes. III. Flächentheorie. Math. Z. **48**, 369–427 (1942). [MF 8718]

[The first two parts had the subtitles "Theorie der Raumkurven" and "Die Flächen konstanter Relativkrümmung $K=rt-s^2$" and appeared in Akad. Wiss. Wien, S.-B. IIa **150**, 1–53 (1941); Math. Z. **47**, 743–777 (1942).] This paper gives a very complete discussion of the theory of surfaces in an isotropic space of three dimensions. Such a space can be considered as a tangential space to the absolute quadric of ordinary four dimensional space; it is a three dimensional space with a metric determined by a pair of intersecting lines in an absolute plane. Its geometry is self dual as to distance and angle, its distance and angle formulas being $d^2=(x_2-x_1)^2+(y_2-y_1)^2$, $\theta^2=(u_2-u_1)^2+(v_2-v_1)^2$ for the distance d of two points (x_1, y_1, z_1), (x_2, y_2, z_2) and the angle θ of two nonisotropic planes $(z=u_1x+v_1y+w_1, z=u_2x+v_2y+w_2)$. Among the many results we may mention the Frenet formulas for curves, the geometry of the "Streifen," consisting of a curve and a set of nonisotropic tangent planes, the theorems of Meusnier and Euler, the existence and properties of lines of curvature, the theorems of Gauss and Codazzi. Minimal surfaces can be introduced after a new definition of surface area. Results are compared to older ones, obtained by Ribaucour, Eiesland, Turrière and E. Müller. Isotropic minimal surfaces turn out to be potential surfaces. *D. J. Struik* (Cambridge, Mass.).

Alexandroff, A. The inner geometry of an arbitrary convex surface. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 467–470 (1941). [MF 9605]

A whole convex surface is the complete boundary of a convex solid in 3-space. A geodesic triangle is normal if each side is the shortest curve joining its end-points. A mid-line of a triangle is a curve of minimum length joining the mid-points of two sides. A convex metric on a 2-dimensional manifold R is one satisfying the three usual conditions and also the following three: (1) for each point x on R and each $\epsilon>0$, there is a neighborhood $V(x, r)$ of x with radius r homeomorphic with part of a plane or cone such that, for any two points y, z of V, the distances ρ, ρ_0 from y to z and from the image of y to the image of z, respectively, satisfy $|\rho-\rho_0|<\epsilon r$; (2) for any two points x, y on R, there is a point z such that $\rho(x, z)=\rho(z, y)=\frac{1}{2}\rho(x, y)$, and such that a change in z, with x fixed, entails a change in y; (3) each point on R has a neighborhood V in which each normal geodesic triangle has mid-lines at least half as long as the corresponding sides. The metric is properly convex if in one triangle in each neighborhood a mid-line is longer than half the corresponding side. A convex metric is complete if it makes R a complete metric space.

Theorem 1. Every whole convex surface possesses a complete convex metric. Theorem 2. For each convex metric given on a sphere, there exists a convex surface having this metric. Theorem 3. For each convex complete metric given on a plane there exists a convex surface having this metric. In these theorems it is necessary to admit regions with boundaries as homeomorphic to spheres if bounded, planes if not bounded. This is accomplished by using two copies, joined along their common boundary. Theorem 5. A properly convex metric may be given only on the following manifolds: a sphere, a plane, a projective plane. Theorem 6. In order that a metric defined by a quadratic form ds^2 with twice continuously differentiable coefficients be convex, it is necessary and sufficient that the Gaussian curvature be positive. Other theorems are given. Proofs are omitted. Some of the results were given in an earlier paper by the same author [same C. R. (N.S.) **30**, 103–106 (1941); these Rev. **2**, 302]. *A. B. Brown* (Flushing, N. Y.).

Wagner, V. On generalized Berwald spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 3–5 (1943). [MF 9574]

A Finsler space is called a generalized Berwald space if the coefficients ${}^*\Gamma^m_{\gamma\beta}$ of the connection defined by Cartan [E. Cartan, Les espaces de Finsler, Hermann, Paris, 1934] are functions of point alone. The author gives a characterization of a two-dimensional generalized Berwald space in terms of the main scalar of the space. *S. Chern.*

Wade, T. L. The factorization of rank tensors. Duke Math. J. **10**, 493–498 (1943). [MF 8971]

Let the numerical tensors $D^{(i)}_{(j)}$ and $E^{(i)}_{(j)}$ in n dimensions be mutually orthogonal idempotents, that is, let $D^{(i)}_{(m)}D^{(m)}_{(j)}=D^{(i)}_{(j)}$, $E^{(i)}_{(m)}E^{(m)}_{(j)}=E^{(i)}_{(j)}$, $D^{(i)}_{(m)}E^{(m)}_{(j)}=E^{(i)}_{(m)}D^{(m)}_{(j)}=0$, $i=i_1, \cdots, i_p$; $j=j_1, \cdots, j_p$. Let the rank of D be s and the rank of E be t. Let

the factors of their rank tensors be given by

$$D^{(i_1)\cdots(i_s)}_{(j_1)\cdots(j_s)} = d^{-1}D^{(i_1)\cdots(i_s)}D_{(j_1)\cdots(j_s)},$$

$$E^{(i_{s+1})\cdots(i_r)}_{(j_{s+1})\cdots(j_r)} = e^{-1}E^{(i_{s+1})\cdots(i_r)}E_{(j_{s+1})\cdots(j_r)},$$

$r=s+t$; d, e nonzero numbers. Then the tensorial factors of the rank tensor of $C^{(i)}_{(j)}=D^{(i)}_{(j)}+E^{(i)}_{(j)}$ are given by

$$C^{(i_1)\cdots(i_r)} = \sum_{\pi} (-1)^{v} D^{(i_{\pi_1})\cdots(i_{\pi_s})}E_{(i_{\pi_{s+1}})\cdots(i_{\pi_r})},$$

$$C_{(j_1)\cdots(j_r)} = \sum_{\tau} (-1)^{w} D_{(j_{\tau_1})\cdots(j_{\tau_s})}E_{(j_{\tau_{s+1}})\cdots(j_{\tau_r})},$$

where v, w are, respectively, the number of transpositions required to bring the π and τ into the natural order 1, 2, $\cdots$, r. Some applications are given for $p=3$, $n=2$ and $p=4$, $n=2$. *D. J. Struik* (Cambridge, Mass.).

NUMERICAL AND GRAPHICAL METHODS

Uhler, H. S. Natural logarithms of small prime numbers. Proc. Nat. Acad. Sci. U. S. A. **29**, 319–325 (1943), and Erratum **30**, 24 (1944). [MF 9446]

Values are given for the natural logarithms of prime numbers from 11 to 101, 9901 and 100! to a large number of decimal places. The number of decimals is 155 for the primes from 41 to 97, 168 for 100! and varies from 290 to 330 for the other primes. The logarithms of primes from 2 to 7 to 273 places have previously been given by J. C. Adams [Proc. Roy. Soc. London **42**, 22–25 (1887)]. *P. W. Ketchum.*

Lowan, Arnold N., Davids, Norman and Levenson, Arthur. Errata to "Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula." Bull. Amer. Math. Soc. **49**, 939 (1943). [MF 9697]

Lowan, A. N. and Hillman, A. A short table of the first five zeros of the transcendental equation $J_0(x)Y_0(kx) - J_0(kx)Y_0(x)=0$. J. Math. Phys. Mass. Inst. Tech. **22**, 208–209 (1943). [MF 9779]

The first five zeros of the above equation were computed for $k=1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4. The zeros are given in the accompanying table I. The products of these zeros by $(k-1)$ are given in table II. The zeros are given to six decimal (seven to eight significant) places. *W. Feller.*

Powell, E. O. An integral related to the radiation integrals. Philos. Mag. (7) **34**, 600–607 (1943). [MF 9332]

The author points out the relation of the integral

$$Rl(x) = \int_1^x \log \xi d\xi/(\xi-1)$$

to problems in astrophysics and quantum mechanics, and because of its importance supplies a table of values. The table gives $Rl(x)$ to seven places of decimals from $x=0$ to $x=6.00$ at intervals of 0.01. Second order central differences are tabulated to assist in interpolation. *W. E. Milne.*

***Anderson, R. L. and Houseman, E. E. Tables of orthogonal polynomial values extended to $N=104$.** Agricult. Exper. Station, Iowa State Coll. of Agricult. Mech. Arts, Statist. Sect., Res. Bull. no. 297, 595–672 (1942).

The set of orthogonal polynomials $\xi'(x)$ satisfying the relation

$$\sum_{x=1}^{n} \xi_i'(x)\xi_j'(x) = 0, \qquad i \neq j,$$

is of fundamental importance in polynomial curve fitting by the method of least squares, and the labor of fitting is greatly reduced when tables of the above polynomials are available. A separate table is required for each value of n. The authors supply tables for each of these orthogonal polynomials from degree 1 to degree 5 inclusive for all integral values of x from 1 to n, and for all integral values of n from 3 to 104. Each polynomial is normalized by choosing the constant factor in such a way that the polynomial always has integral values for integral values of x. The values of $\sum_{x=1}^{n} \xi_i'^2(x)$ are supplied in all cases.

W. E. Milne (Corvallis, Ore.).

Satakopan, V. The advantages in using orthogonalised terms in a polynomial for curve-fitting. Indian J. Phys. **17**, 115–120 (1943). [MF 9093]

S. M. Kerawala [Indian J. Phys. **15**, 241–276 (1941); these Rev. **3**, 276] discussed the coefficients in general polynomials, of lower than the sixth degree, fitted by least squares to a function $y(x)$ known for n equally spaced values of x. To describe the six least squares polynomials approximating to a given function, 21 coefficients were needed. Satakopan here points out the well-known circumstance that the statistically independent least squares values of the first six A's in the representation $\sum_0^5 A_i\varphi_i(x)$ involving the Tchebycheff orthogonal polynomials $\varphi_i(x)$ suffice to determine all the 21 coefficients. He gives equations for finding the six A's from the 21 coefficients, and the 21 coefficients from the six A's. Another advantage of employing orthogonal polynomials is the resulting ease of computing the sum of squares of residuals. That the greatest practical advantage is the ease with which statistically independent terms $A_i\varphi_i(x)$ may be removed successively from $y(x)$, until the residuals are satisfactorily small, is not pointed out. Nor does the article mention the tables reviewed above. *T. E. Sterne* (Aberdeen, Md.).

de Jekhowsky, Benjamin. Formules de contrôle et moyen d'augmenter la précision du calcul des déterminants de la méthode de Lagrange-Andoyer pour la détermination d'une orbite képlérienne. C. R. Acad. Sci. Paris **215**, 220–222 (1942). [MF 9489]

Wright, L. T., Jr. The solution of simultaneous linear equations by an approximation method. Cornell Univ. Engrg. Exper. Station. Bulletin no. **31**, 6 pp. (1943). [MF 9334]

Numerical method for approximate solution of n nonhomogoneous linear equations in n unknowns with real numerical coefficients. The procedure is applied to two problems, each for $n=4$. Let the equations be

$$F_j = \sum a_{ij}x_i + a_j = 0.$$

The a_j are initial values of the F_j corresponding to all $x_i=0$. Values of the x_i are then systematically selected which will make the absolute greatest F_j come closer to zero. If this approximation is sufficient for the purpose in hand, the work stops. If the F_j are not all close enough to zero, the process is repeated as often as required. For the examples chosen, the reviewer does not recognize a gain over standard methods of reducing the matrices to normal

forms. The process converges about one additional decimal for each step in the problem worked out in detail. The author refers to his method as an extension of one developed in Southwell [Relaxation Methods in Engineering Science, Oxford University Press, New York, 1940; these Rev. **3**, 152]. *A. J. Kempner* (Boulder, Colo.).

Fischer, C. H. The rate of interest in instalment contracts. Amer. Math. Monthly **50**, 545–547 (1943). [MF 9530]

Eichler, Martin. Zur numerischen Lösung von Gleichungen mit reellen Koeffizienten. J. Reine Angew. Math. **184**, 124–128 (1942). [MF 8984]

From the equation with real coefficients

$$(1) \qquad f(x)=a_0+a_1x+\cdots+a_nx^n=a_n\cdot\prod(x+r_\nu), \qquad n>1,$$

is derived

$$(2) \qquad F(x)=\text{const.}\cdot\prod\{x+(r_\nu+r_\mu)/2\}, \qquad \nu<\mu,$$

which has a real zero for every nonreal zero of (1). "Zur Herleitung des Verfahrens . . . verstehe ich unter k den Körper der reellen Zahlen . . . . Schliesslich sei $K=k(a_0, a_1, \cdots, a_n)$. Jetzt hat $F(x)$ Koeffizienten in K, wenn die noch offene Konstante von (2) in K liegt, und ist in K irreduzibel." The statement that (2) is irreducible in K is not clear to the reviewer. This statement is used in the following manner: $F(x)$ is transformed into a determinant (4) which, expanded, is a polynomial in x of degree at most $n(n-1)/2$ [given as $n(n+1)/2$ in the text] with coefficients in K. The author then gives the degree of (2) in x as $n(n-1)/2$ [see above bracketed comment] and concludes in familiar manner that, since (4) is not of higher degree than (2) and (2) is irreducible, therefore (4) is, except for a possible constant factor, identically equal to (2). Regardless of the validity of the argument, the method of the paper may be applicable in a wide range of cases. The idea of using (2) for the computation of complex roots of equations with real coefficients is not new. The author presents his method as one "das auch die Auffindung der komplexen Wurzeln recht einfach macht." No comparison with any of the many known methods is offered.

A. J. Kempner (Boulder, Colo.).

Hansel, C. W. Graphical differentiation and integration. Philos. Mag. (7) **34**, 565–574 (1943). [MF 9136]

The author describes various constructions which may be applied to empirical data for the approximate determination of derivatives, integrals, empirical algebraic equations and empirical differential equations. The central idea is to carry out the computations on a nomogram in such a way that averages may be found by visual inspection. Thus, in the case of empirical equations, the data are plotted as index lines on trial nomograms. Approximate coincidence of all these lines indicates the discovery of an appropriate type of equation, and the equation itself is obtained by estimating the centroid of the actual intersections. The author gives detailed directions for performing these constructions, together with 18 examples, but omits any consideration of their validity. [The reviewer remarks that taking centroids on a nomogram will lead to different results than are obtained by the usual averaging processes in rectangular coördinates, especially near certain values of the variables. In particular, the procedure for differentiation fails for slopes near -1. This is not a serious defect in this case, but the analytical bases for the author's methods need elucidation.] *P. W. Ketchum* (Urbana, Ill.).

Heinhold, J. Zur mechanischen Integration von Differentialgleichungen. Z. Instrumentenkunde **63**, 71–74 (1943). [MF 8808]

A mechanical integrator is described which consists of a sphere constrained entirely by six sharp edged wheels rolling in contact with it at the two polar points and four symmetrically placed equatorial points. The axes of the equatorial wheels are fixed in position and direction and parallel to a direction D in the equatorial plane. The axes of the polar wheels are fixed in position and are parallel to each other and to the equatorial plane, but makes a variable angle with D. This angle is controlled so as always to equal arc tan $f(x)$; this can be effected directly from the curve for $f(x)$ by means of a simple linkage. If one of the equatorial wheels is rotated through an angle x, the other equatorial wheels will rotate through angles proportional to $\int f(x)dx$, x and $\int(1/f(x))dx$, respectively. The main advantage claimed for the integrator is the absence of slipping of the point of contact of the wheels with the sphere. Hence a larger torque output can be obtained than can be had from the table type integrators ordinarily used in differential analyzers. Thus a differential analyzer using these integrators could solve simple differential equations without torque amplifiers, or at least with a minimum of torque amplification.

P. W. Ketchum (Urbana, Ill.).

Kormes, Mark. Numerical solution of the boundary value problem for the potential equation by means of punched cards. Rev. Sci. Instruments **14**, 248–250 (1943). [MF 8923]

In this paper is described a punched-cards calculation method of obtaining numerical solutions of the finite difference equations which are used to approximate to the two-dimensional Laplace equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$. The method is suitable for use with standard I.B.M. or Remington Rand equipment. A reasonable set of values is assumed to start, and successive approximations equivalent to replacing the values are used. Successive approximations are computed in this way and the process is repeated until two successive calculations differ only in decimal places beyond the significant figure. For the example described 145 net points were employed and the tenth approximation gave an accuracy to two decimal places. The machine time required per approximation is estimated under thirty minutes. *H. Poritsky* (Schenectady, N. Y.).

Fischer, T. Abgleichung des Polarplanimeters. Allg. Vermessgs.-Nachr. **54**, 197–199 (1942). [MF 9346]

Donnell, L. H. A chart for plotting relations between variables over their entire real range. Quart. Appl. Math. **1**, 276–277 (1943). [MF 9369]

The chart "consists of an ordinary Cartesian plot over the range -1 to $+1$ of each variable, with adjoining Cartesian plots of the reciprocals of the variable from 0 to -1 and from $+1$ to $0\cdots$." *W. E. Milne.*

Neder, Ludwig. Über die Berechnung von beliebig langen Aggregaten und "Faktorgregaten" mit Hilfe eines einzigen einfachen Nomogramms. Z. Angew. Math. Mech. **22**, 238 (1942). [MF 8922]

By a "Faktorgregat" the author understands expressions of the form $f_1(x_1)f_2(x_2)\cdots f_n(x_n)$.

Pinkwart. Die Umformung gleichartiger Koordinaten. I, II, III. Allg. Vermessgs.-Nachr. **54**, 97–110, 121–133, 209–223, 234–243, 253–265 (1942). [MF 9344]

König. Vereinfachte Schnittpunktberechnung mit der Rechenmaschine. Allg. Vermessgs.-Nachr. **54**, 226–228 (1942). [MF 9348]

Stummerer. Ein weiterer Beitrag zur Lösung der Hansenschen Aufgabe. Allg. Vermessgs.-Nachr. **54**, 61 (1942). [MF 9343]

MATHEMATICAL PHYSICS

March, A. Raum, Zeit und Naturgesetze. Z. Phys. **119**, 100–101 (1942). [MF 9414]

Cf. the author's paper in Z. Phys. **117**, 413–416 (1941); these Rev. **4**, 96.

Darrow, Karl K. The new statistical mechanics. Bell System Tech. J. **22**, 362–392 (1943). [MF 9554]

Mahan, Archie I. A mathematical proof of Stokes' reversibility principle. J. Opt. Soc. Amer. **33**, 621–626 (1943). [MF 9408]

G. G. Stokes formulated in 1849 the following principle: if any material system, in which the forces acting depend only on the position of the particles, is in motion and if at any instant the velocities of the particles are reversed, the previous motion will be repeated in reverse order. The author investigates the validity of this principle for optical problems. Applying Fresnel's formulae to refraction and reflection at a plane, he computes amplitude and phase of a well defined state after reversion and compares them with amplitude and phase of the incident wave. He concludes that they will be identical only under conditions which are stated in the paper (identical phase change and reflection from the two reversing surfaces and neglect of absorption).

M. Herzberger (Rochester, N. Y.).

Linfoot, E. H. On some optical systems employing aspherical surfaces. Monthly Not. Roy. Astr. Soc. **103**, 210–221 (1943). [MF 9247]

The "see-saw diagram" method of C. R. Burch [Monthly Not. Roy. Astr. Soc. **102**, 159 (1942)] is applied to obtain general formulas for the Seidel errors, excluding distortion, of optical systems consisting of two mirrors and a nearly plane parallel plate. It is shown that the only flat-fielded anastigmatism with both mirrors spherical is equivalent to the field-flattened Schmidt, while, in general, flat-fielded anastigmat requires asphericity of at least two of the optical surfaces. Nevertheless, special consideration is given to the case where both mirrors are spherical and only the correcting plate is figured. A system is designed suitable for use as an astrographic camera. Although this system is not strictly anastigmatic, it is shown that its field curvature can be satisfactorily controlled. An estimation is given of the performance of the system as compared with that of Schwarzschild's two mirror aplanats. *P. Boeder.*

Korringa, J. Classification of aberrations in rotationally symmetrical optical systems. Physica **8**, 477–496 (1941). [MF 9542]

Herzberger's formulas of image errors of the fifth order [J. Opt. Soc. Amer. **29**, 395–406 (1939); these Rev. **1**, 24, 349] are investigated to obtain an insight into the structure of the relation between the aberrations and to simplify the transformation. For this purpose those linear combinations of third and fifth order aberrations are determined which transform easily when object and stop are moved simultaneously or separately. For the third order aberrations this reduction yields the well-known fact that the Petzval sum is invariant. For the fifth order, three quantities are obtained which are said to constitute the first group of Smith. Five further quantities in the third order and seven in the fifth order constitute the zero group of Smith. The remaining quantities of the fifth order are not transformed because they are expressions in the third order aberrations. In addition to the transformation formulas, the aberrations of one single refractive surface and the formulas for the composition of optical systems are derived. Finally the process of actual calculation is indicated. *P. Boeder.*

Herzberger, M. Gaussian optics and Gaussian brackets. J. Opt. Soc. Amer. **33**, 651–655 (1943). [MF 9670]

Gaussian brackets are applied to paraxial systems of any number of surfaces to relate changes in thickness, index and radius to variations in focal length, back focus and magnification. Brackets of construction elements are set up from which are obtained the a, b, c and d of the fundamental equations $x'=ax+b\xi$, $\xi'=cx+d\xi$, where $ad-bc=1$. Two typical modifications of optical systems making compensations indicated by this method are considered. Finite ray image errors in the form recommended by the author in earlier papers [Trans. Amer. Math. Soc. **53**, 218–229 (1943); Quart. Appl. Math. **1**, 69–77 (1943); these Rev. **4**, 204] can be expressed with the help of these brackets.

P. Boeder (Southbridge, Mass.).

Flint, H. T. The fundamental equation of quantum mechanics. Philos. Mag. (7) **34**, 496–502 (1942). [MF 8881]

The results of previous work on parallel displacement of matrices in five-dimensional spaces are used to write an invariant second order five-dimensional wave equation for an external electromagnetic field. The nonrelativistic approximation of this equation is compared to the Schrödinger equation and it is found that, in addition to the spin terms in the energy, there are other terms. No quantitative comparison of the solutions of this equation with those of the Schrödinger equation is given; hence it is not possible to judge this equation in light of the experimental evidence. The reviewer does not agree with the statement [cf. pp. 500, 501] that the second order equations should be regarded as fundamental in the theory since the usual objections to an equation of second order in the time would seem to apply here. *A. H. Taub* (Princeton, N. J.).

de Beauregard, Olivier Costa. Sur dix relations conséquences des équations de Dirac. C. R. Acad. Sci. Paris **214**, 818–820 (1942). [MF 9455]

The author gives ten relations that are consequences of Dirac's equations. Physical interpretations for the relations that are new are given. *M. Wyman* (Edmonton, Alta.).

Haenzel, G. Die de Brogliesche Theorie des Photons in geometrischer Darstellung. Z. Tech. Phys. **24**, 87–90 (1943). [MF 9736]

In three previous papers [Jber. Deutsch. Math. Verein. **49**, 215–242 (1940); **50**, 121–129 (1940); **52**, 103–117 (1942); these Rev. **1**, 262; **2**, 152; **5**, 9] the author constructed geometrical models representing one and two electron systems. In a similar way, the author now represents the theory of photons by geometric models.

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